

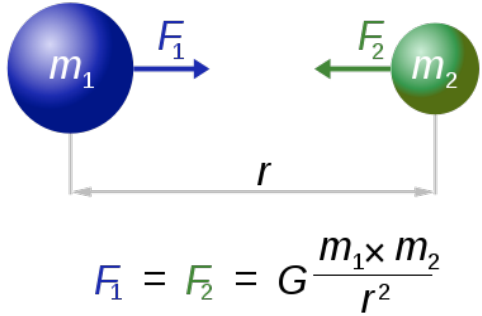
Gravitational Waves

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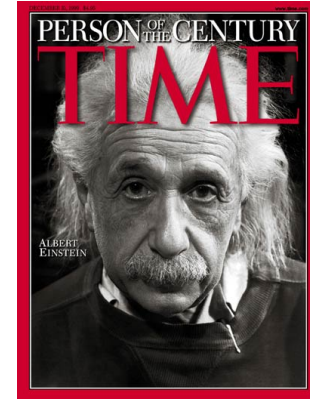


From Newton to Einstein

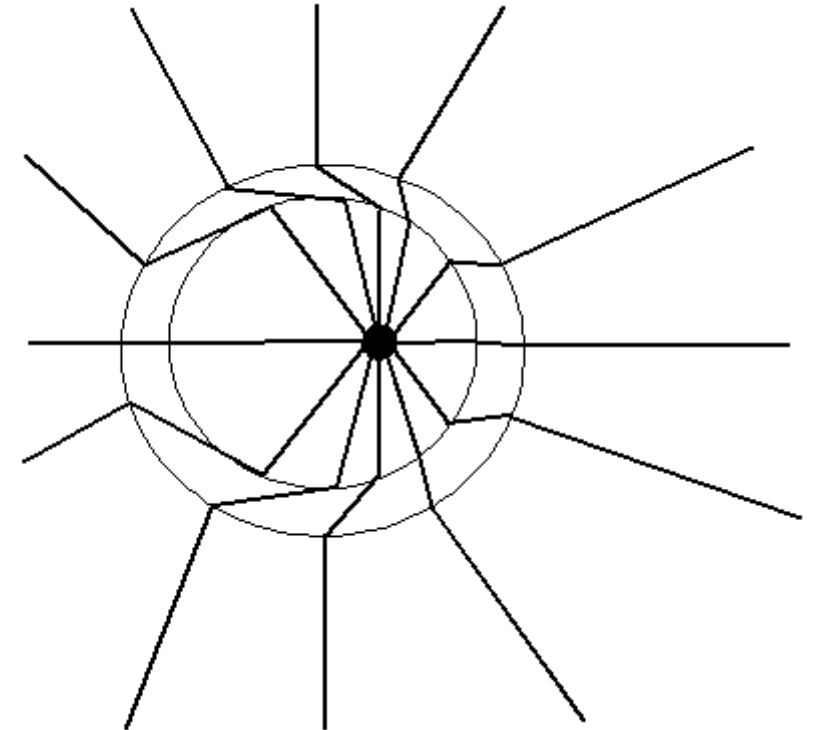


$$\begin{aligned} m\ddot{\vec{x}} &= -m\vec{\nabla}\phi \\ \vec{F} &= -m\vec{\nabla}\phi \\ \Delta\phi &= 4\pi G\rho(\vec{x}, t) \end{aligned}$$

Newton's
field
equation



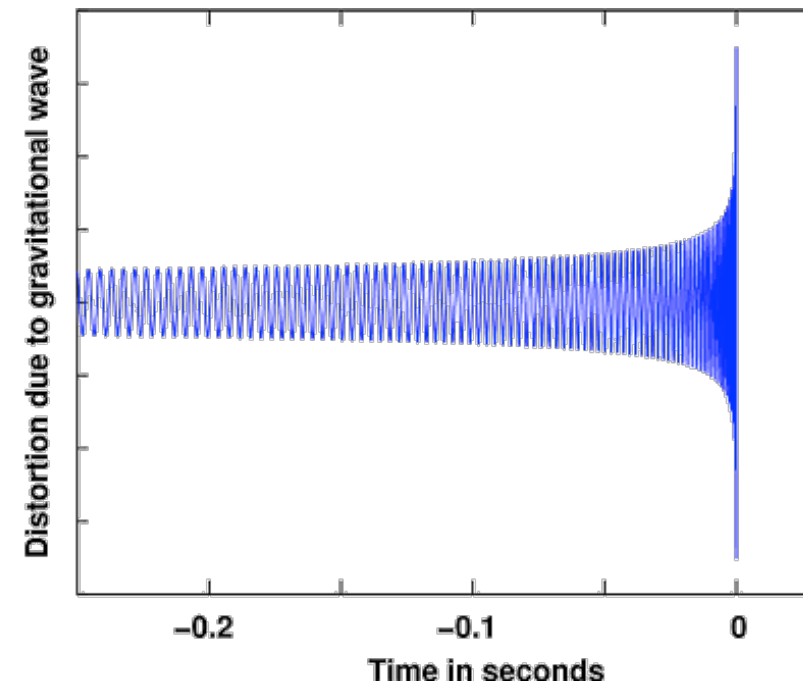
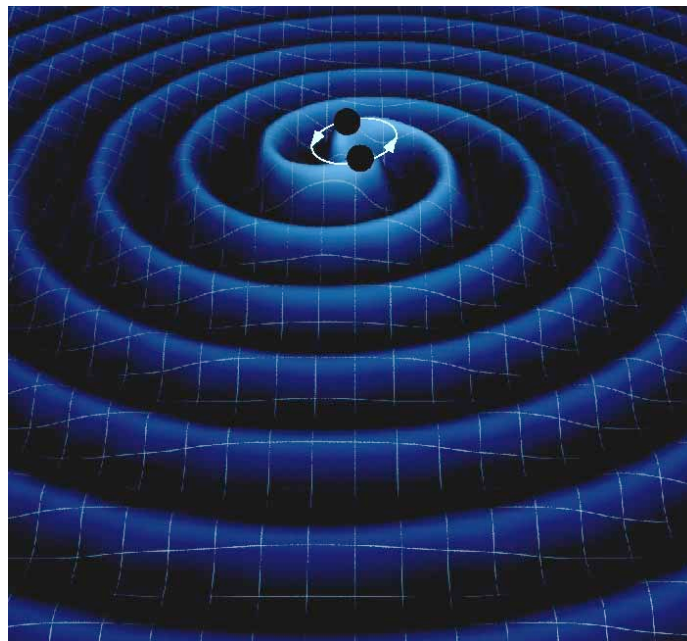
- 1905: **special relativity**: Speed of light is the same for all inertial observers, information travels at finite speed.
- => Need to change our theory of gravity!
 - 1915: theory of **general relativity - Einstein field equations**
- Accelerated electric charges: at larger distance from the charge the field lines get updated later - outward propagating disturbance = EM wave.
- Finite propagation speed in GR -> expect GW.



Listening to Gravitational Waves

- **Observation of electromagnetic waves:**

- Incoherent superposition of waves from many particles -> loss of phase information
- observe intensity and spectrum depending on direction -> image of the source



- The **GW signal carries information about the bulk motion of objects:**
~ listening to the soundtrack of the universe.
- **Measure phase+amplitude -> sensitivity $\sim 1/r$.**
- For binary systems the **GW signal** encodes the masses, spins, sky location + possibly new physics - **needs decoding.**
- **GW detectors are more antennas/microphones than telescopes.**

B L A C K H O L E H U N T E R

How to find the signals: play @ blackholehunter.org



Welcome to Black Hole Hunter

Your objective is to listen to the signal below, and then determine which of the four data files contains the signal you heard. To begin with, the signals will be relatively loud. As you progress through the game the signals will get quieter.



SIGNAL



DATA 1



DATA 2



DATA 3



DATA 4



MASS 1 = 1.2 $M_{\odot}$
MASS 2 = 1.2 $M_{\odot}$
INCLINATION = 10°

	X	Y	Z
SPIN 1 =	0,	0,	0.8
SPIN 2 =	0,	0,	0



How to automate finding events?



Don't know the signal: look for excess power, correlated between detectors.

When you do know/guess:

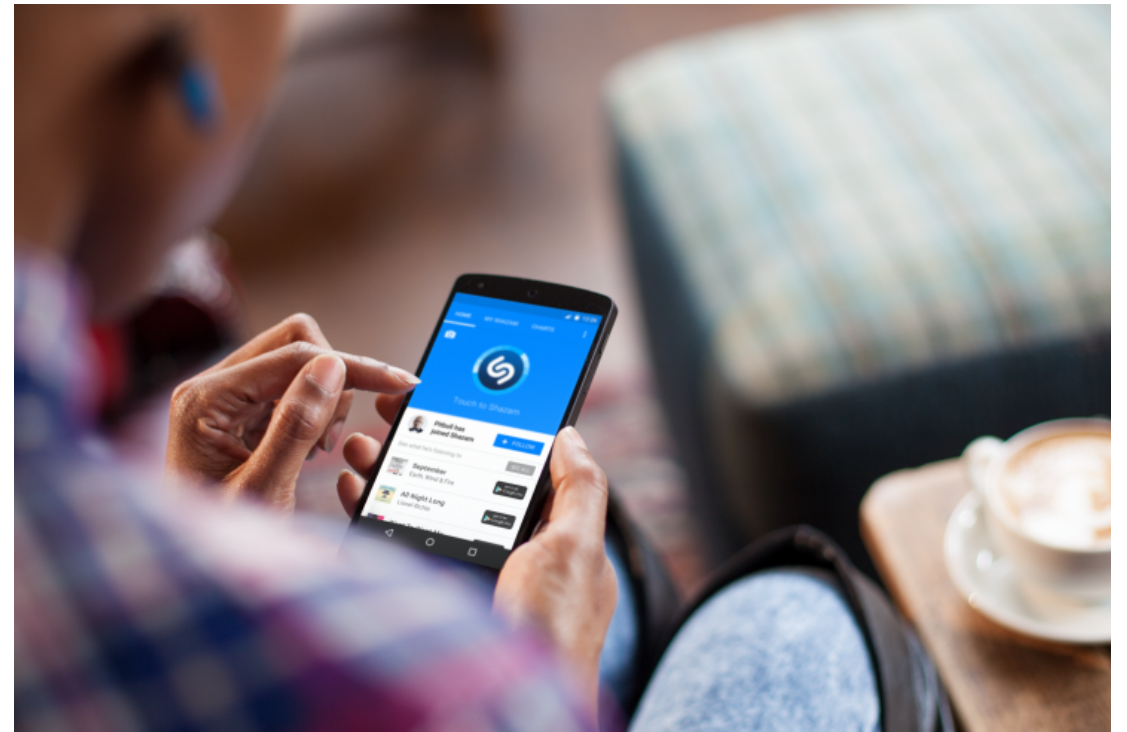
Compare

data = noise + signal
with a data base of signals.

Basic idea ~Shazam etc.

Need:

- data base of signals computed from GR
- Systematic way of comparing



Geometric units

- $G = c = 1$
- Can measure time, length in units of mass.
- Examples:
 - Mass to length scale:
Schwarzschild radius $R_S = 2 GM/c^2$
 - Characteristic gravitational time scale for an object:
 GM/c^3 - higher mass corresponds to lower frequency.
- Maximal radiated power scale $1M/1M$.

Geometric units

Can we distinguish a GW signal
from a redshifted GW signal?

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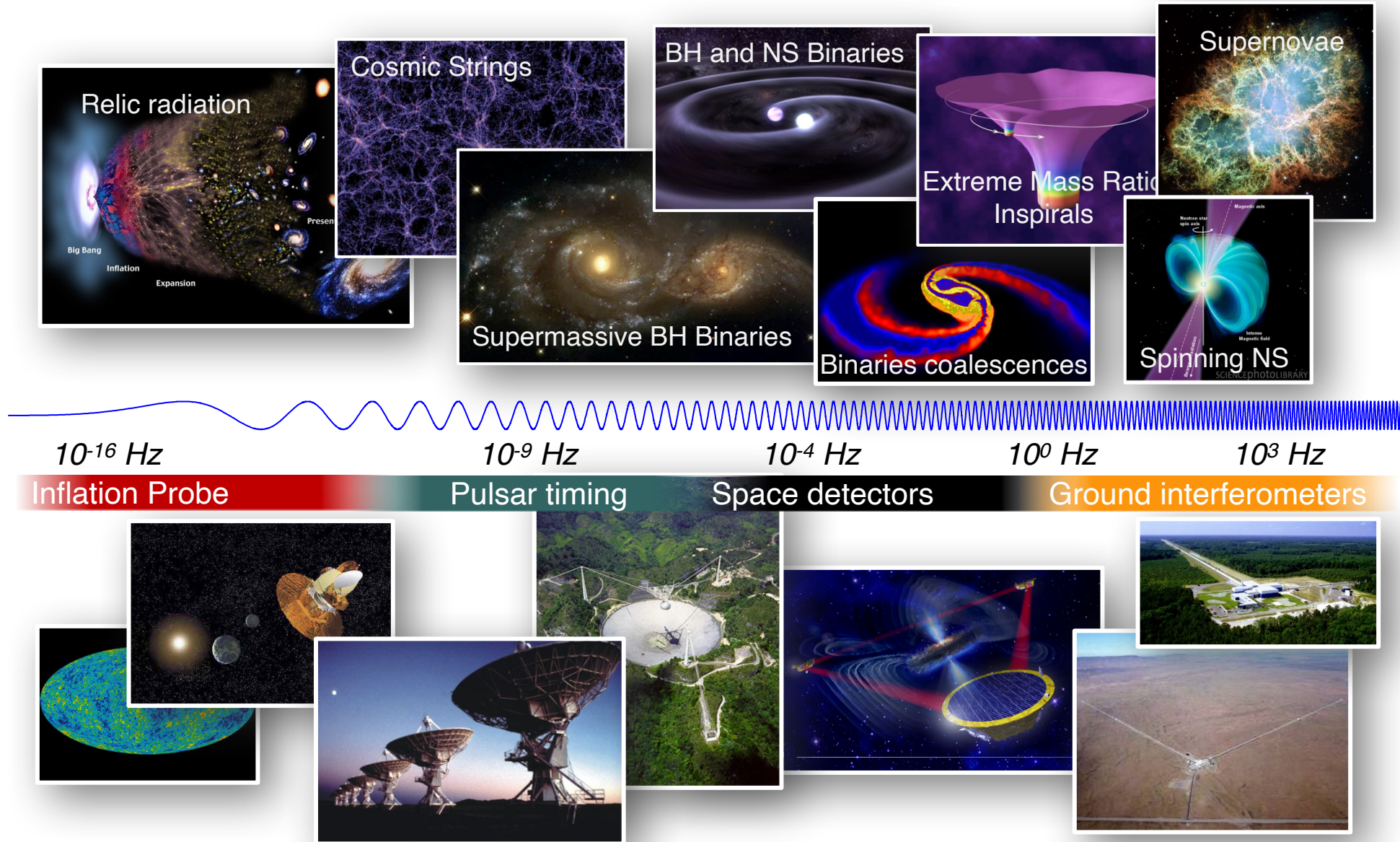
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Can we distinguish a GW signal from a redshifted GW signal?

Not for BHs - can measure luminosity distance, not Redshift.

For redshift we need EM counterpart.

Gravitational Wave Spectrum

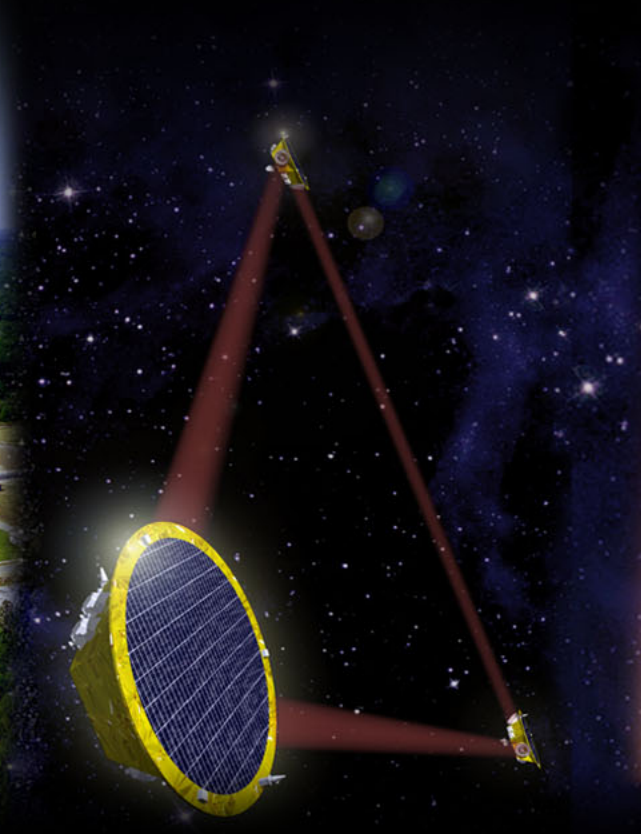


Gravitational Wave Periods

Milliseconds



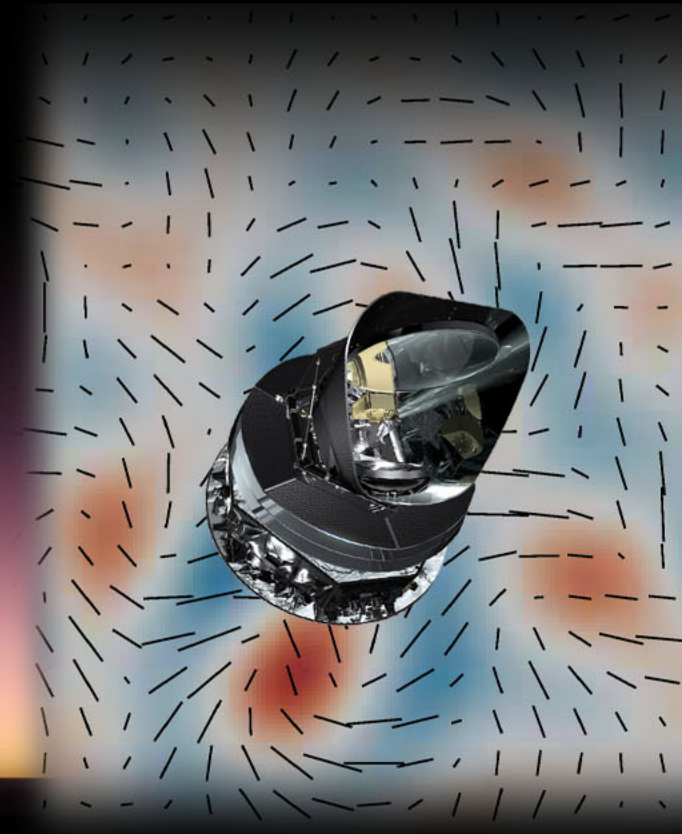
**Minutes
to Hours**



**Years
to Decades**



**Billions
of Years**



Gravity is geometry

- Guiding principle: equivalence principle.
- Freely falling objects correspond to a special class of curves: Geodesics = straightest possible = extremal length
- The mathematics: differential geometry - tensor fields on manifolds
- The fundamental object: the metric tensor

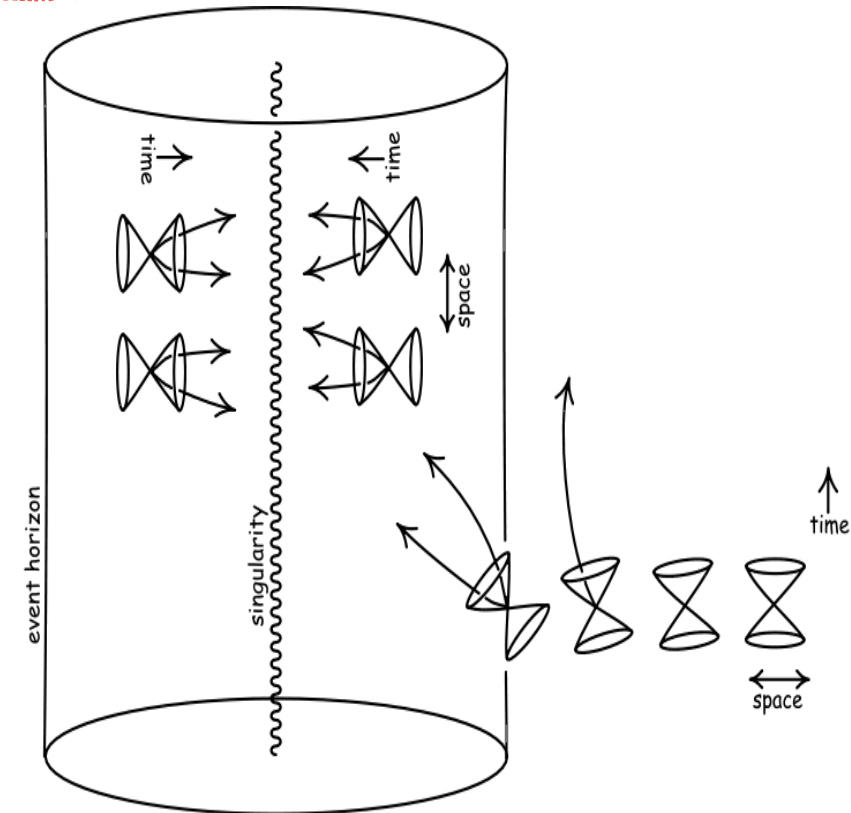
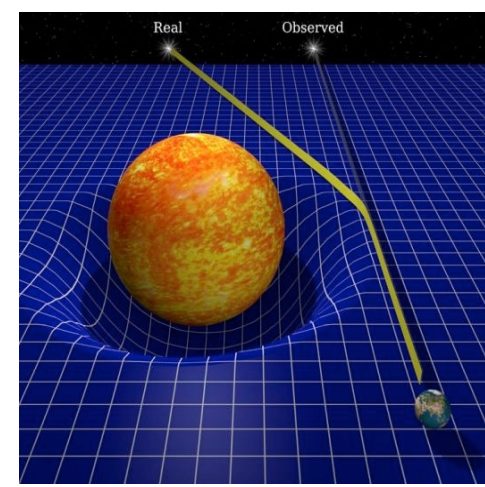
$$g_{ab} : \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- Need field equations: PDEs for the metric - Einstein equations:

Spacetime curvature

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

Energy momentum tensor of matter

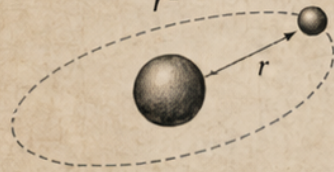


KNOWN TERRITORY

Newtonian gravity.
Linearized gravitational waves.

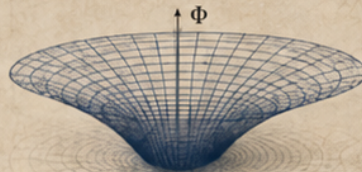
1. NEWTON'S LAW

$$F = G \frac{m_1 m_2}{r^2}$$



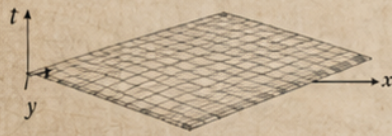
2. POISSON'S EQUATION

$$\nabla^2 \Phi = 4\pi G \rho$$



3. WEAK-FIELD METRIC

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$



4. LINEARIZED WAVE EQUATION

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

1. EINSTEIN FIELD EQUATIONS

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

2. LINE ELEMENT

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

3. CONNECTIONS

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$$

4. CURVATURE

$$R^\alpha_{\beta\mu\nu}$$

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

5. NONLINEAR WAVE COUPLINGS

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$+ N_{\mu\nu} [h, h]$$

$$+ O(h^3)$$

self-interaction
& mode coupling

6. STRONG-FIELD PHENOMENA

- Black holes
- Mergers
- Ringdown
- Tidal effects

Here be dragons

DRAGON TERRITORY

General relativity.
Nonlinear gravitational waves.

Follow the standard textbook strategy to explain gravitational waves: use the linearised Einstein equations (linearised around flat space) as much as possible - this is not fundamental, we are just short on time!

Linearised Einstein equations: setup

BACKGROUND & INVERSE METRIC

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$$

EQUIVALENCE PRINCIPLE $\Rightarrow$ NO LOCAL GW ENERGY

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \quad \partial_\alpha g_{\mu\nu}(p) = 0$$

LINEARIZED EINSTEIN TENSOR (MESSY!)

$$G_{\mu\nu}^{(1)} = \frac{1}{2} (\partial_\rho \partial_\mu h^\rho{}_\nu + \partial_\rho \partial_\nu h^\rho{}_\mu - \square h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \square h)$$

TRACE-REVERSED PERTURBATION

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}, \quad \bar{h} = -h$$

HARMONIC GAUGE $\sim$ LORENZ GAUGE

$$\partial^\mu \bar{h}_{\mu\nu} = \partial^\mu h_{\mu\nu} = 0$$

$\Rightarrow$ WAVE EQUATION

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

$\Rightarrow$ Gravitational waves travel at the speed of light.

Interlude: electromagnetism as a gauge theory

FIELD STRENGTH & MAXWELL EQUATIONS

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu, \quad \partial_{[\lambda} F_{\mu\nu]} = 0$$

LORENZ GAUGE $\Rightarrow$ WAVE EQUATION

$$\partial_\mu A^\mu = 0 \Rightarrow \quad \square A^\mu = 4\pi J^\mu$$

RETARDED FIELD

$$A^\mu(t, \mathbf{x}) = \int d^3x' \frac{J^\mu(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

POLARISATIONS: E_x, E_y

- Maxwell's equation in covariant form.
- Wave equation after imposing gauge condition.
- Construction of wave solution with Green's function.
- In a multipole expansion the leading term is the dipole.
There is no monopole radiation.
- 2 polarisations: orthogonal transversal E/B fields.

From harmonic to transverse–traceless gauge

- Use superposition of plane wave solutions with wave vector $\mathbf{k}$ to construct general solution.
- Use remaining gauge freedom to adopt the TT-gauge: traceless and transverse purely spatial strain tensor.

PLANE WAVE & HARMONIC GAUGE CONSTRAINT

$$\bar{h}_{\mu\nu} = \Re\{A_{\mu\nu} e^{ik_\alpha x^\alpha}\}, \quad k^\alpha k_\alpha = 0$$

$$k^\mu A_{\mu\nu} = 0$$

THE THREE TT CONDITIONS

$$h_{0\mu}^{\text{TT}} = 0, \quad \partial^i h_{ij}^{\text{TT}} = 0, \quad h^{\text{TT} i}{}_i = 0$$

TRANSVERSE–TRACELESS PERTURBATION (WAVE ALONG +Z)

$$h_{ij}^{\text{TT}}(t - z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

TT LINE ELEMENT

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij}^{\text{TT}}(t - z)) dx^i dx^j$$

- Degrees of freedom count (universal for gauge theories as constrained Hamiltonian systems):
- Physical = all - constraints - gauge
 = all - 2 x gauge
 2 = 10 - 4 - 4

Effect on test masses: geodesic deviation

- **Tidal effect** — neighbouring free masses obey the geodesic-deviation equation.
- **Detector frame** — with $u \approx (1,0,0,0)$ only $R_{0i0j}^{(1)}$ matters.

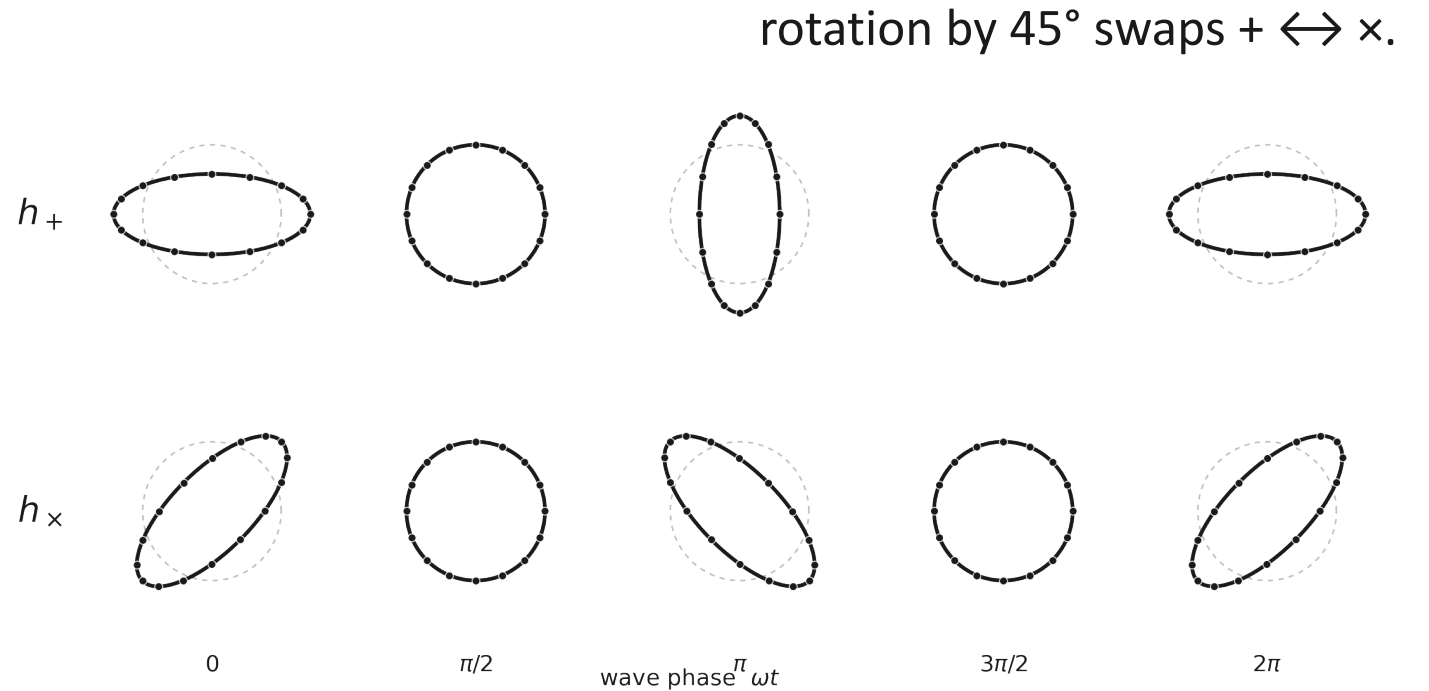
GEODESIC DEVIATION & CURVATURE

$$\frac{d^2 \xi^\mu}{d\tau^2} = -R^\mu_{\alpha\beta\gamma} u^\alpha u^\beta \xi^\gamma$$

$$R_{0i0j}^{(1)} = -\frac{1}{2} \partial_t^2 h_{ij}^{\text{TT}}$$

TIDAL ACCELERATION IN TT GAUGE

$$\frac{d^2 \xi^i}{dt^2} = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} \xi^j$$



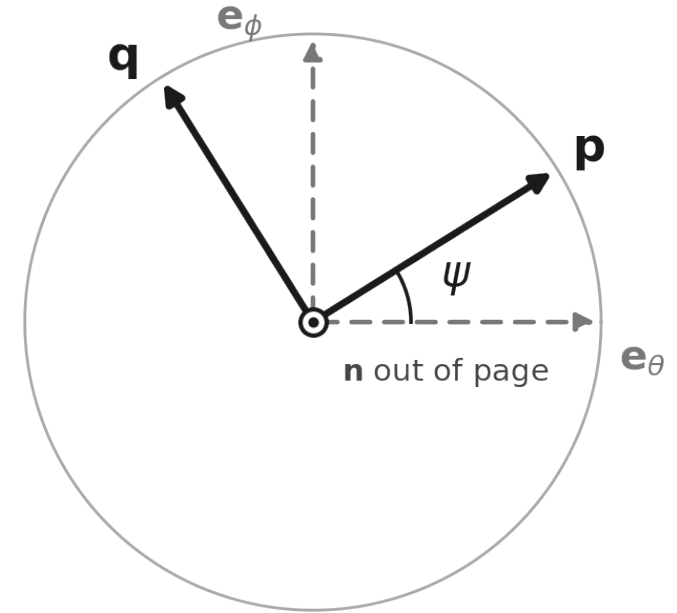
A ring of free test masses deformed by + and $\times$ over one wave period (wave into the page).

DISPLACEMENT: STRAIN RESPONSE ALONG A DIRECTION $\hat{n}$

$$\frac{\delta \ell(t)}{\ell_0} = \frac{1}{2} h_{ij}^{\text{TT}} \hat{n}^i \hat{n}^j$$

Polarization basis & the polarization angle ψ

- **Basis tensors** — e_+ and $e_\times$ are built from orthonormal vectors p, q spanning the plane transverse to the propagation direction n .
- **Polarization angle ψ** — ψ rotates (p, q) about n — it fixes the orientation of the $+$ and $\times$ axes on the sky.



POLARIZATION DECOMPOSITION

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times$$

POLARIZATION BASIS TENSORS

$$e_{ij}^+ = p_i p_j - q_i q_j, \quad e_{ij}^\times = p_i q_j + q_i p_j$$

BASIS ROTATED BY THE POLARIZATION ANGLE ψ

$$\mathbf{p} = \mathbf{e}_\theta \cos \psi + \mathbf{e}_\phi \sin \psi, \quad \mathbf{q} = -\mathbf{e}_\theta \sin \psi + \mathbf{e}_\phi \cos \psi$$

COMPLEX GRAVITATIONAL-WAVE STRAIN

$$h(t) \equiv h_+(t) - i h_\times(t)$$

BASIS ROTATION BY $\psi \rightarrow$ A PHASE (SPIN WEIGHT -2)

$$\text{rotate basis by } \psi : h_+ - i h_\times \longrightarrow e^{2i\psi} (h_+ - i h_\times)$$

Detector response for ground-based interferometers

MEASURED STRAIN = WEIGHTED SUM OF POLARIZATIONS

$$h(t) = F_+ h_+(t) + F_\times h_\times(t)$$

Need > 1 detector to measure both polarisations.

SYMBOLS

θ, ϕ — source direction in the detector frame

ψ — polarization angle

DETECTOR TENSOR & PROJECTION

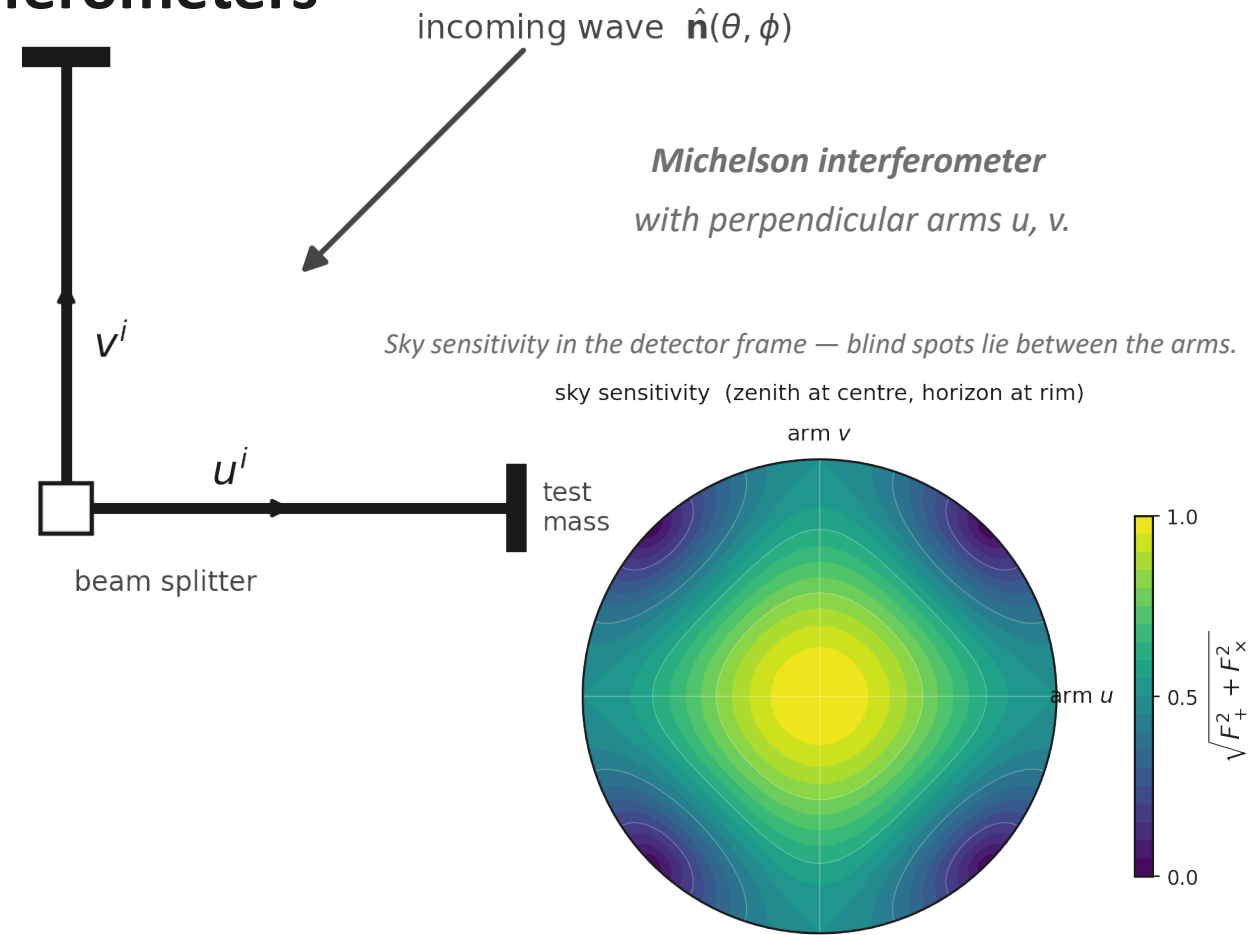
$$h(t) = D^{ij} h_{ij}(t), \quad D^{ij} = \frac{1}{2} (u^i u^j - v^i v^j)$$

$$F_+ = D^{ij} e_{ij}^+, \quad F_\times = D^{ij} e_{ij}^\times$$

ANTENNA RESPONSE FUNCTIONS

$$F_+ = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi$$

$$F_\times = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi$$



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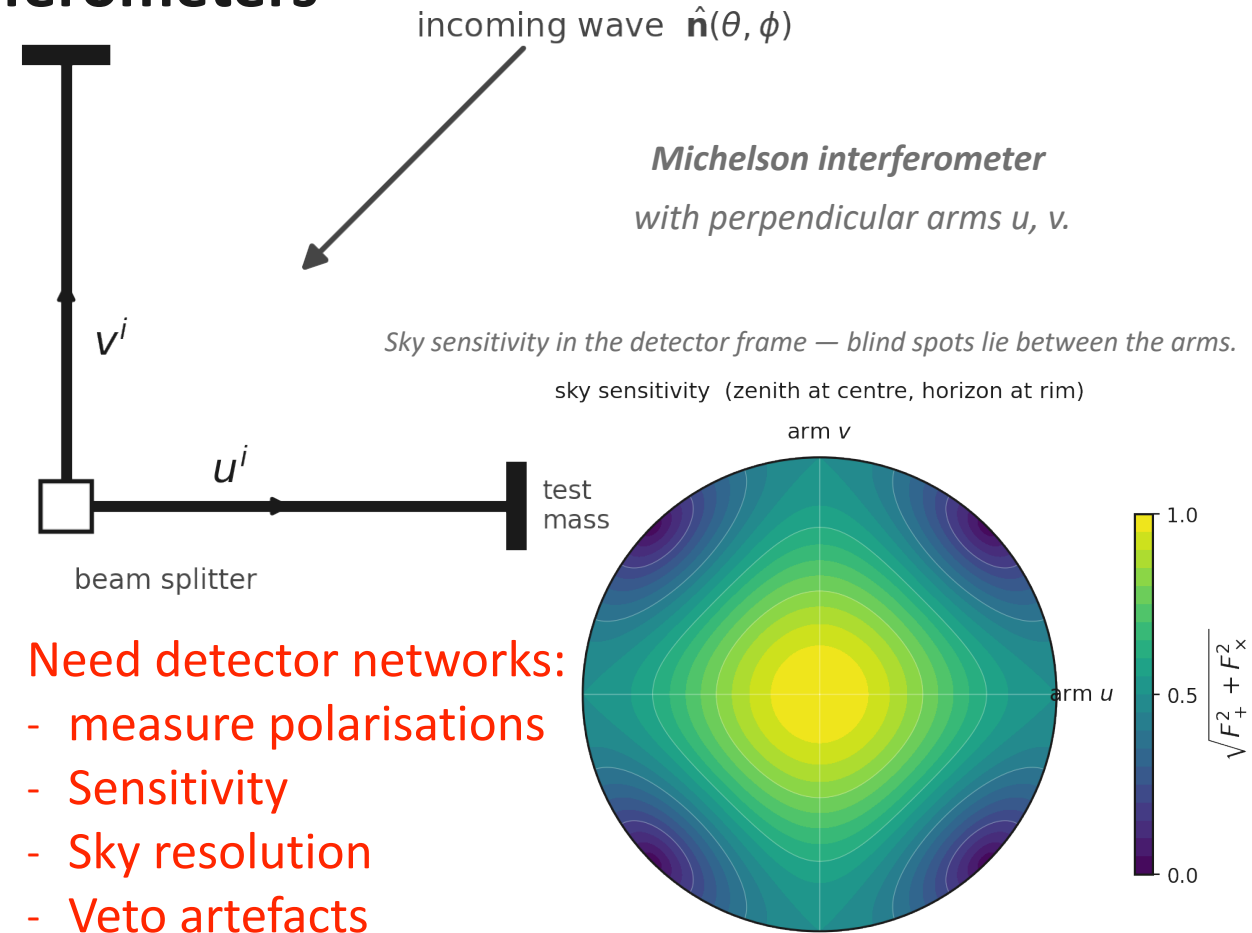
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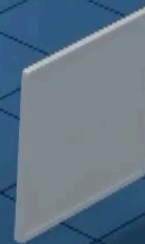
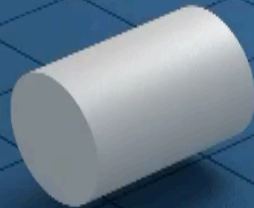


Need detector networks:

- measure polarisations
- Sensitivity
- Sky resolution
- Veto artefacts
- Uptime

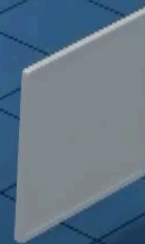
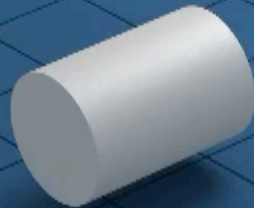
Interferometry – The scheme

Credit: LIGO/T. Pyle



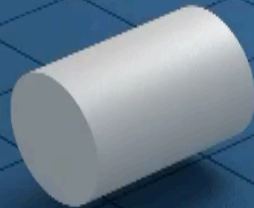
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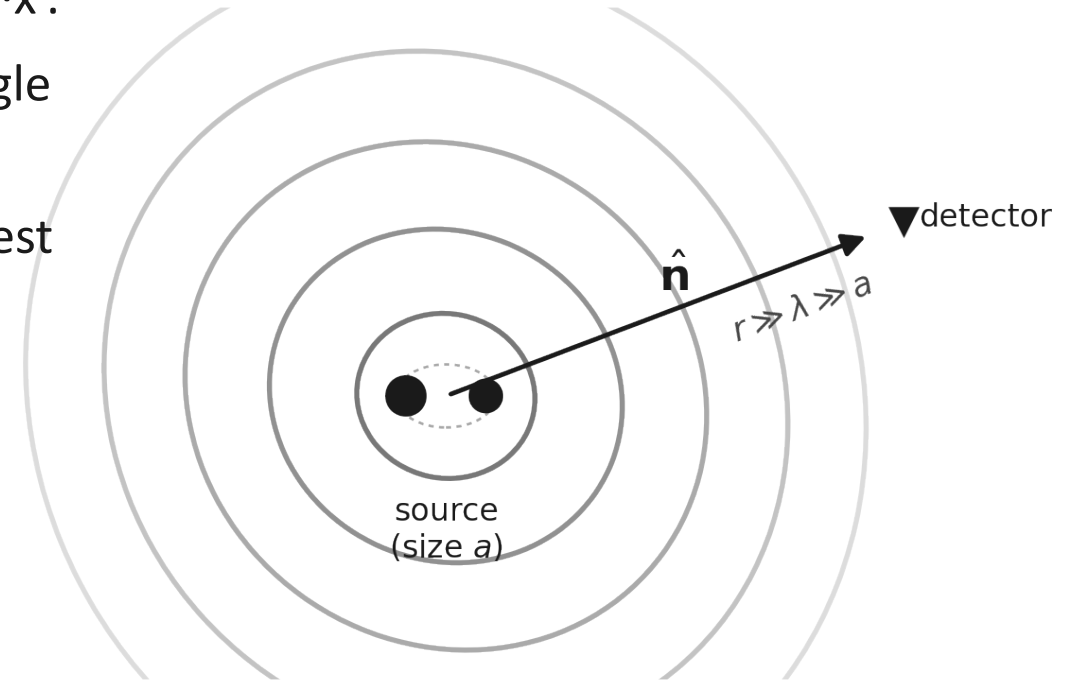
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GW emission: leading order approximation for qualitative understanding

- Far zone — for an observer at $r \gg$ source size, $|\mathbf{x} - \mathbf{x}'| \approx r - \mathbf{n} \cdot \mathbf{x}'$.
- Long wavelength — for $\lambda \gg a$, evaluate the source at the single retarded time $t - r$.
- Leading term — the radiative field is set by the source's lowest surviving multipole
- Slowly moving sources: $v \ll c$.
- Weak gravitational field - drop quadratic and higher terms.



REMEMBER WAVE EQUATION IN HARMONIC GAUGE

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

RETARDED GREEN'S FUNCTION & FAR-ZONE EXPANSION

$$\square G_{\text{ret}} = \delta^{(4)}(x - x'), \quad G_{\text{ret}} = \frac{\delta(t' - t + |\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{n} \cdot \mathbf{x}' + \mathcal{O}(a^2/r), \quad \mathbf{n} = \mathbf{x}/r$$

RETARDED SOLUTION

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3x' \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Result: the quadrupole formula

MASS QUADRUPOLE & CONSERVATION IDENTITY

$$I^{ij}(t) = \int d^3x \rho x^i x^j, \quad Q_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I^k_k$$
$$\ddot{I}^{ij}(t) = 2 \int d^3x T^{ij}(t, \mathbf{x})$$

THE QUADRUPOLE FORMULA

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{c^4 r} \frac{d^2 Q_{ij}^{\text{TT}}(t - r/c)}{dt^2}$$

① EFFECTIVE STRESS-ENERGY

$$\langle T_{\mu\nu}^{\text{GW}} \rangle = \frac{1}{32\pi G} \langle \partial_\mu h_{ij}^{\text{TT}} \partial_\nu h_{ij}^{\text{TT}} \rangle$$

② ANGULAR AVERAGE

$$\int d\Omega \ddot{Q}_{ij}^{\text{TT}} \ddot{Q}_{ij}^{\text{TT}} = \frac{4\pi}{5} \ddot{Q}_{ij} \ddot{Q}_{ij}$$

③ QUADRUPOLE LUMINOSITY

$$P = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle$$

Spacetime is very stiff: Need very large changes in the quadrupole moment to deform the metric!

Which type of astrophysical processes will be efficient?

Quadrupole approximation for circular binary

CIRCULAR-BINARY POLARIZATIONS

$$h_+(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3} \frac{1 + \cos^2 \theta}{2} \cos(2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi),$$

$$h_\times(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3} \cos \theta \sin(2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi),$$

CIRCULAR FLUX

$$\frac{dE}{dt} = -\frac{32}{5} \frac{G^4}{c^5} \eta^2 \left(\frac{v}{c} \right)^{10} (1 + O(v^2) + \dots) \quad v = (GM\omega)^{1/3}$$

CONVERT $R \leftrightarrow \omega$ with Kepler formula

$$\omega^2(R) = \frac{GM}{R^3} (1 + O((v/c)^2))$$

CHIRP MASS AND SYMMETRIC MASS RATIO

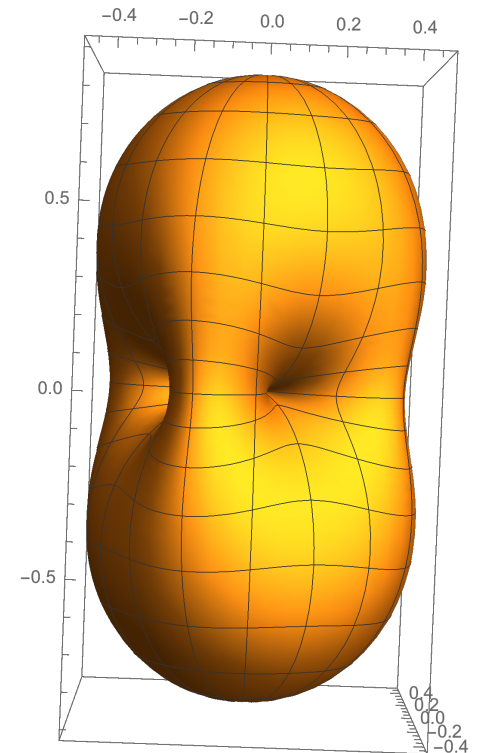
$$\mathcal{M} = \eta^{3/5} M, \quad \eta = \frac{m_1 m_2}{M^2} \in (0, \frac{1}{4}]$$

$f_{\text{GW}} = 2 f_{\text{orb}} + \text{higher order}$

Attention with f vs. ω ,
factors of 2 and π !

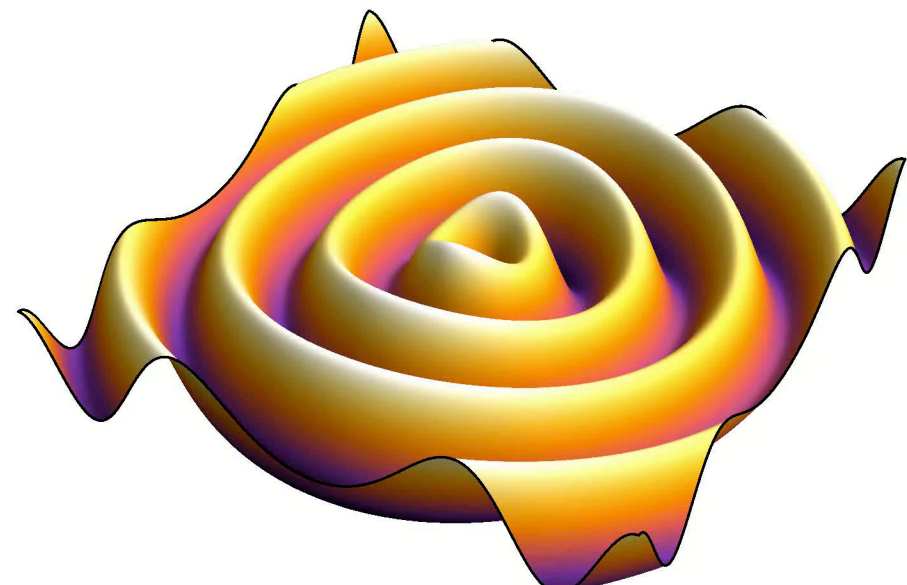
**maximise
emission:**

- Face on/off
- Large mass
- High freq./small separation



Quadrupole wave pattern

Gravitational waves from orbital motion



Radiation power of the sun (electromagnetic).

$4 \times 10^{26} \text{ W}$

Iron bar (1000 t, 100 m, 3 Hz)

10^{-26} W

Earth in orbit around sun

200 W

Stellar binary system

$10^{15} - 10^{30} \text{ W}$

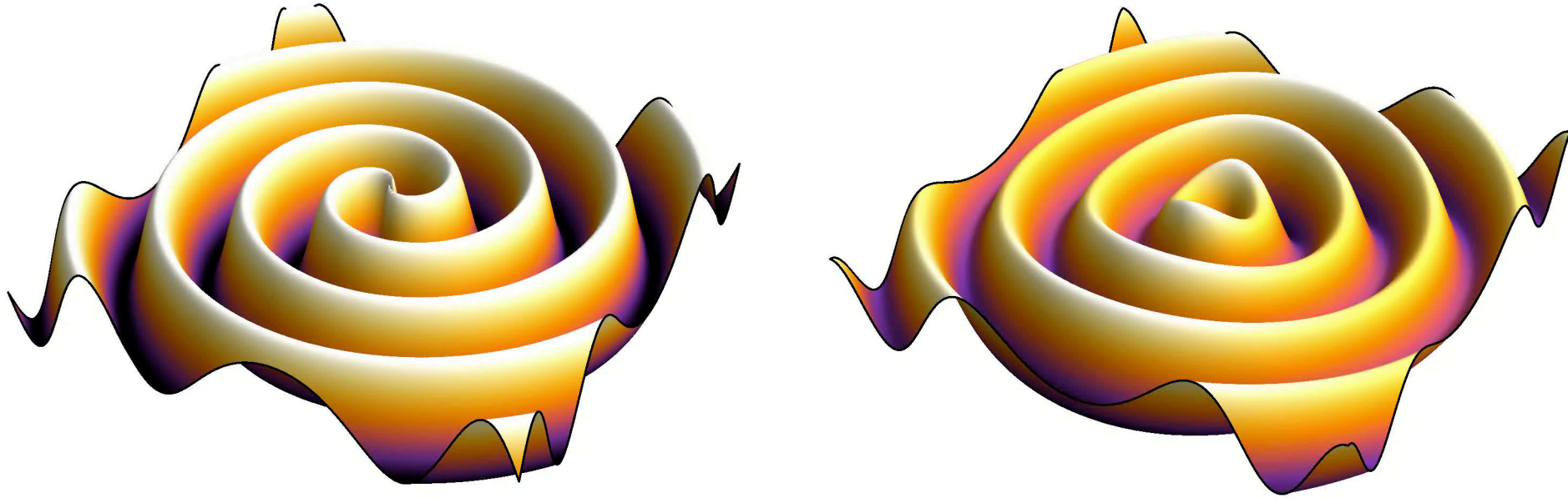
BNS system (100 km, 100 Hz)

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GW150914 maximal

$3.6 \times 10^{49} \text{ W}$

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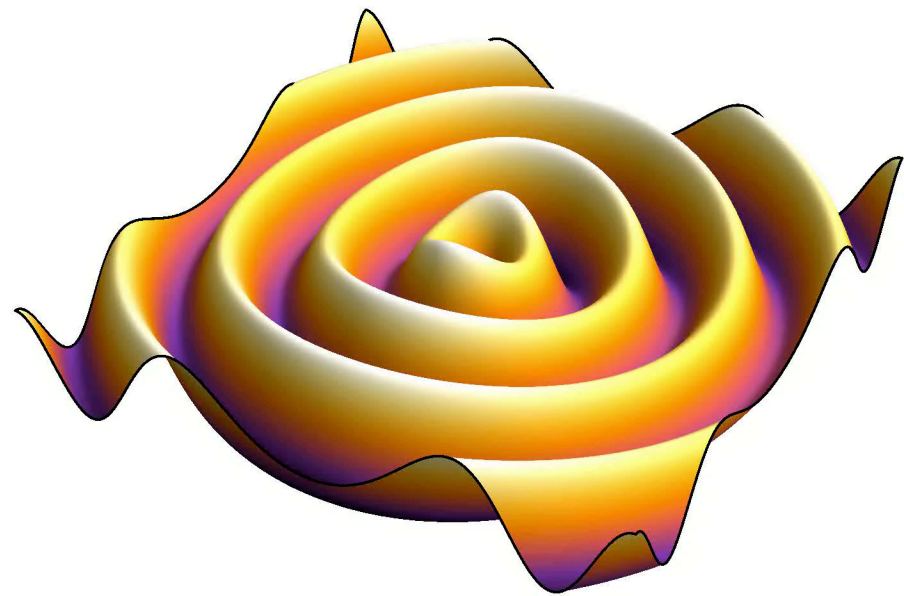
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How does this scale with mass?

$3.6 \times 10^{49} \text{ W}$

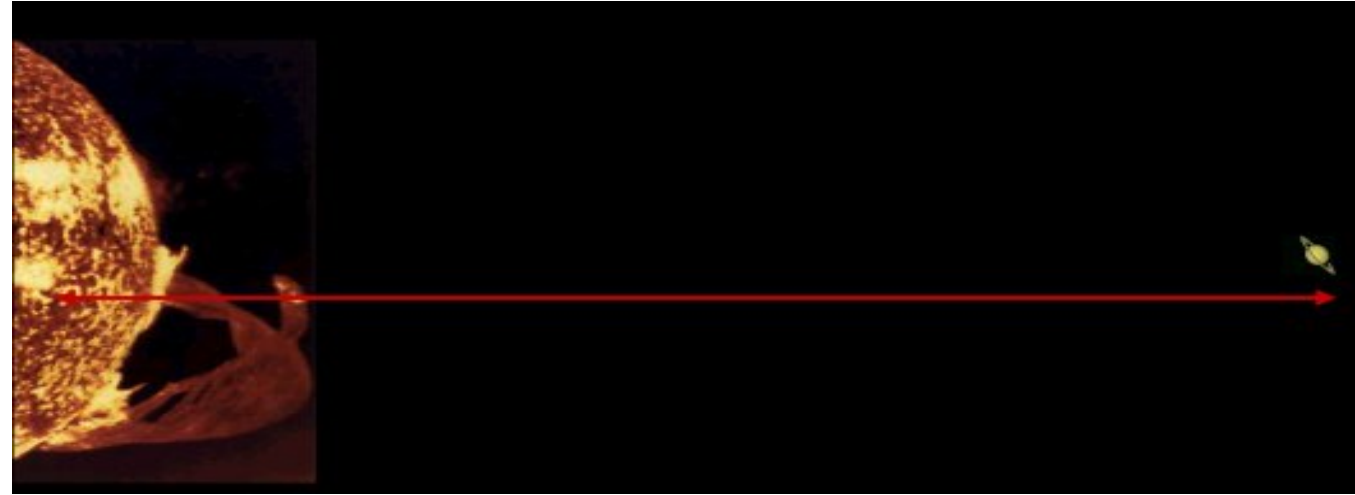
How Small is the Stretching and Squeezing?

The amount of stretching and squeezing of space which is **predicted** to occur near the Earth due to events such as the coalescence of a pair of neutron stars within about 100 million light-years from Earth is about one part in 10^{22} .

For 4 km arm length that is a relative length change of 4×10^{-19} .

In order to identify astrophysical signals in the noise, and to decode their sources, we need to

- Build extremely sensitive detectors.
- Understand the plausible signals.
- Understand the noise.



Equivalent to detecting change in the motion of Saturn by the diameter of a single hydrogen atom!

What do we need to predict typical signal amplitudes?

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TABLE V: Detection rates for compact binary coalescence sources.

IFO	Source ^a	$\dot{N}_{\text{low}}$ yr^{-1}	$\dot{N}_{\text{re}}$ yr^{-1}	$\dot{N}_{\text{high}}$ yr^{-1}	$\dot{N}_{\text{max}}$ yr^{-1}
Initial	NS-NS	2×10^{-4}	0.02	0.2	0.6
	NS-BH	7×10^{-5}	0.004	0.1	
	BH-BH	2×10^{-4}	0.007	0.5	
	IMRI into IMBH			$< 0.001^b$	0.01^c
	IMBH-IMBH			10^{-4d}	10^{-3e}
Advanced	NS-NS	0.4	40	400	1000
	NS-BH	0.2	10	300	
	BH-BH	0.4	20	1000	
	IMRI into IMBH			10^b	300^c
	IMBH-IMBH			0.1^d	1^e

Equivalent to detecting change in the motion of Saturn by the diameter of a single hydrogen atom!

What do we need to predict typical signal amplitudes?

Derive the Chirp - first step beyond Newton

Simplest approximation of the signal: Newtonian point particles + energy loss from quadrupole formula:

- Start with energy as function of separation R or orbital frequency ω :
- Convert between ω and R with Kepler: $\omega^2 R^3 = G M$.
- Compute energy loss $P = -dE/dt$ at leading order with quadrupole formula.
- Compute the rate of change of any quantity X (e.g. $X = \omega, R, \dots$) as

$$E(R) = m_1 + m_2 - M \frac{\eta}{2} \frac{M}{R}$$

$$E(\omega) = m_1 + m_2 - M \frac{\eta}{2} \left(\frac{(M\omega)^2}{G} \right)^{\frac{1}{3}}$$

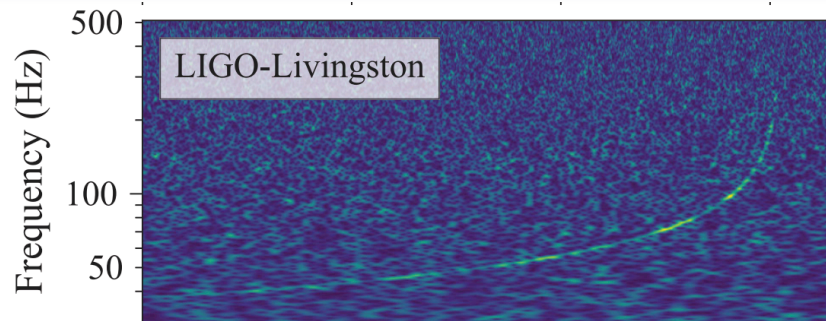
$$\frac{dX}{dt} = \frac{\frac{dE}{dt}}{\frac{dE}{dX}}$$

=> FREQUENCY EVOLUTION — SET BY $\mathcal{M}$ ALONE

$$f(t) = \frac{1}{\pi} \left[\frac{5}{256 (t_c - t)} \right]^{3/8} \left(\frac{GM}{c^3} \right)^{-5/8}$$

$$R(t) = \left(\frac{256}{5} \eta M^3 \right)^{\frac{1}{4}} (t_c - t)^{\frac{1}{4}}$$

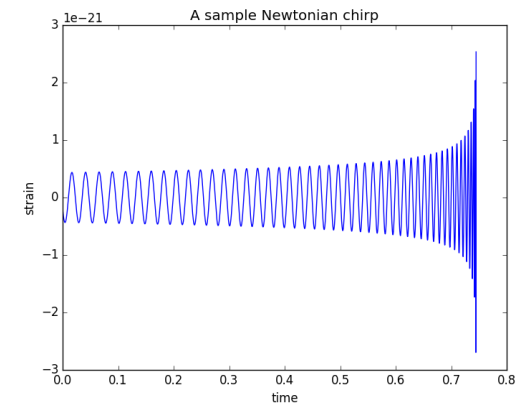
$$t_c - t = \frac{5}{256} (\pi f)^{-8/3} \mathcal{M}^{-5/3}$$



LSC+Virgo, PRL 119 161101 (2017)

For GWs we also need phase.

$$\frac{d\phi}{dt} = \omega$$



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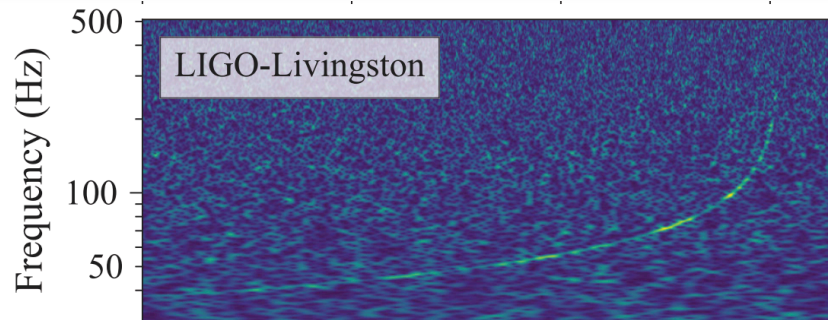
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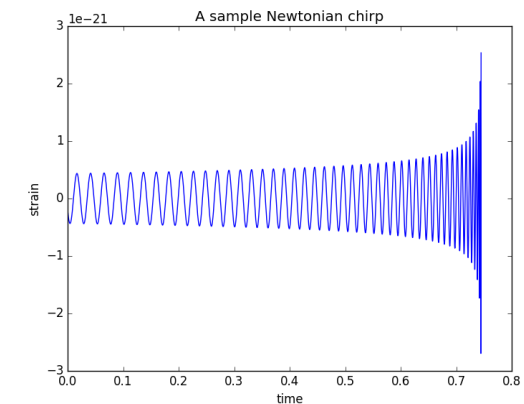
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For GWs we also need phase.

$$\frac{d\phi}{dt} = \omega$$



Degeneracy:

only measure M_c , not both component masses.

Good for searches - bad for parameter estimation!

The simplicity of black holes in GR

- **How many equilibrium black hole solutions exist?**

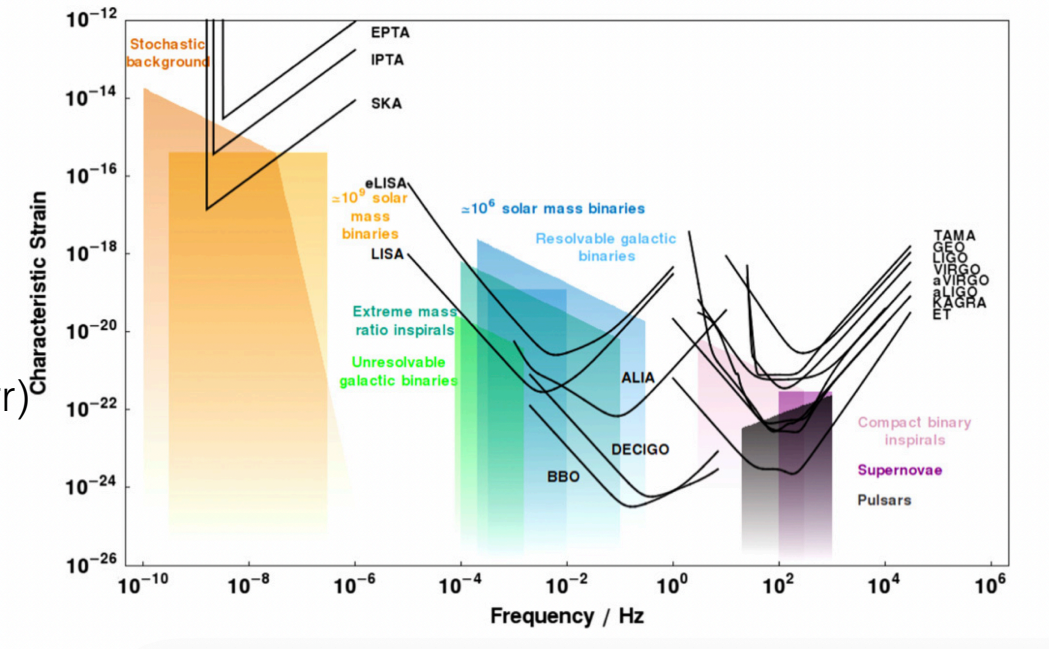
Consider fundamental long range interactions - GR + EM field.

- **Black hole uniqueness theorem / no hair theorem:**

- All time independent solutions belong to Kerr-Newman family: parameterised by mass M , angular momentum J , charge Q .
- Astrophysical bodies are $\sim$ electrically neutral $\Rightarrow$ **M, J** . (Kerr)
- Cosmic censorship conjecture: dimensionless **$J/M^2 < 1$** . (equality: extreme Kerr)

- **Parameter space of binary black holes:**

- **Quasi-circular** (circular at infinite separation) - initial momentum is tuned to such orbits.
 - 2 masses + 2 spin vectors = 8 parameters
 - **total mass = scale parameter, since G and c are the only fundamental constants in GR)**
model 7 - measure 8.
- **Eccentric:** 2 more. (but eccentricity is shed rapidly during inspiral)
- $\Rightarrow$ **9 intrinsic parameters**
- **7 extrinsic parameters:** coalescence time, location in the sky (distance, sky position), orientation (inclination, reference phase, polarisation).



Eccentric binaries: evolution & circularization

- **Peters ODEs** — orbit-averaged $\dot{a}$ and $\dot{e}$ govern the inspiral at quadrupole order.
- **Circularization** — $\dot{e} < 0$: gravitational radiation sheds eccentricity (Peters–Mathews).
- **Invariant** — the Peters constant c_0 gives an implicit relation $e(a)$.
- **Observations** — **expect** most binaries to enter the LVK band nearly circular.

PETERS EVOLUTION EQUATIONS

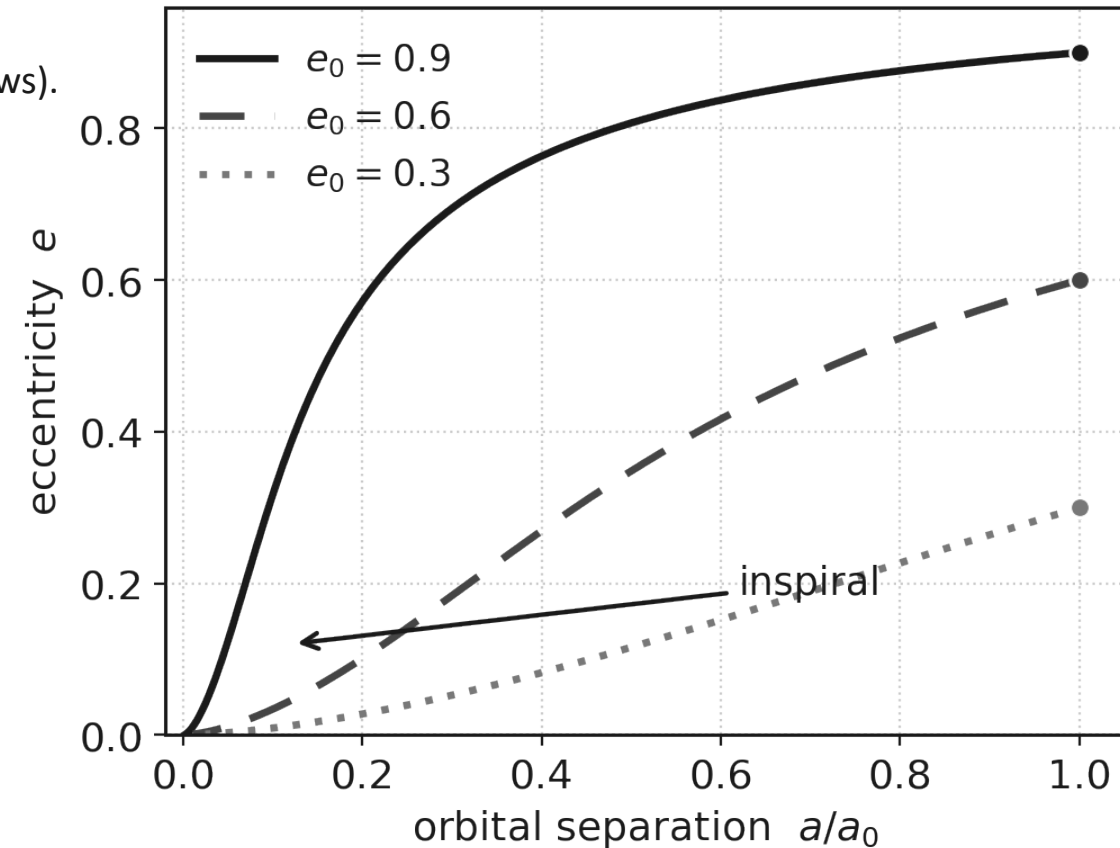
$$\dot{a} = -\frac{64}{5} \frac{G^3}{c^5} \frac{\mu M^2}{a^3} \quad \dot{e} = -\frac{304}{15} \frac{G^3}{c^5} \frac{\mu M^2}{a^4} \frac{e \left(1 + \frac{121}{304} e^2\right)}{(1 - e^2)^{5/2}}$$

PETERS CONSTANT (INVARIANT)

$$c_0 = a (1 - e^2) e^{-12/19} \left(1 + \frac{121}{304} e^2\right)^{-870/2299}, \quad \dot{c}_0 = 0$$

ORBIT-AVERAGED GW FLUX

$$\langle \dot{E}_{\text{GW}} \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{\mu^2 M^3}{a^5} \underbrace{\frac{1 + \frac{73}{24} e^2 + \frac{37}{96} e^4}{(1 - e^2)^{7/2}}}_{F(e)}$$



Inspiral drives $e \rightarrow 0$; tracks shown for several initial eccentricities.

What could be the interpretation of observing eccentric binaries?

Interior Schwarzschild solution

- spherical symmetry + time independent + matter, e.g. EOS $P(\rho)$
- Interior solution with ideal fluid in equilibrium: TOV equation

$$ds^2 = e^\nu c^2 dt^2 - \left(1 - \frac{2Gm}{rc^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\frac{d\nu}{dr} = - \left(\frac{2}{P + \rho c^2} \right) \frac{dP}{dr} \qquad m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$$

$$\frac{dP}{dr} = - \frac{Gm}{r^2} \rho \left(1 + \frac{P}{\rho c^2} \right) \left(1 + \frac{4\pi r^3 P}{mc^2} \right) \left(1 - \frac{2Gm}{rc^2} \right)^{-1}$$

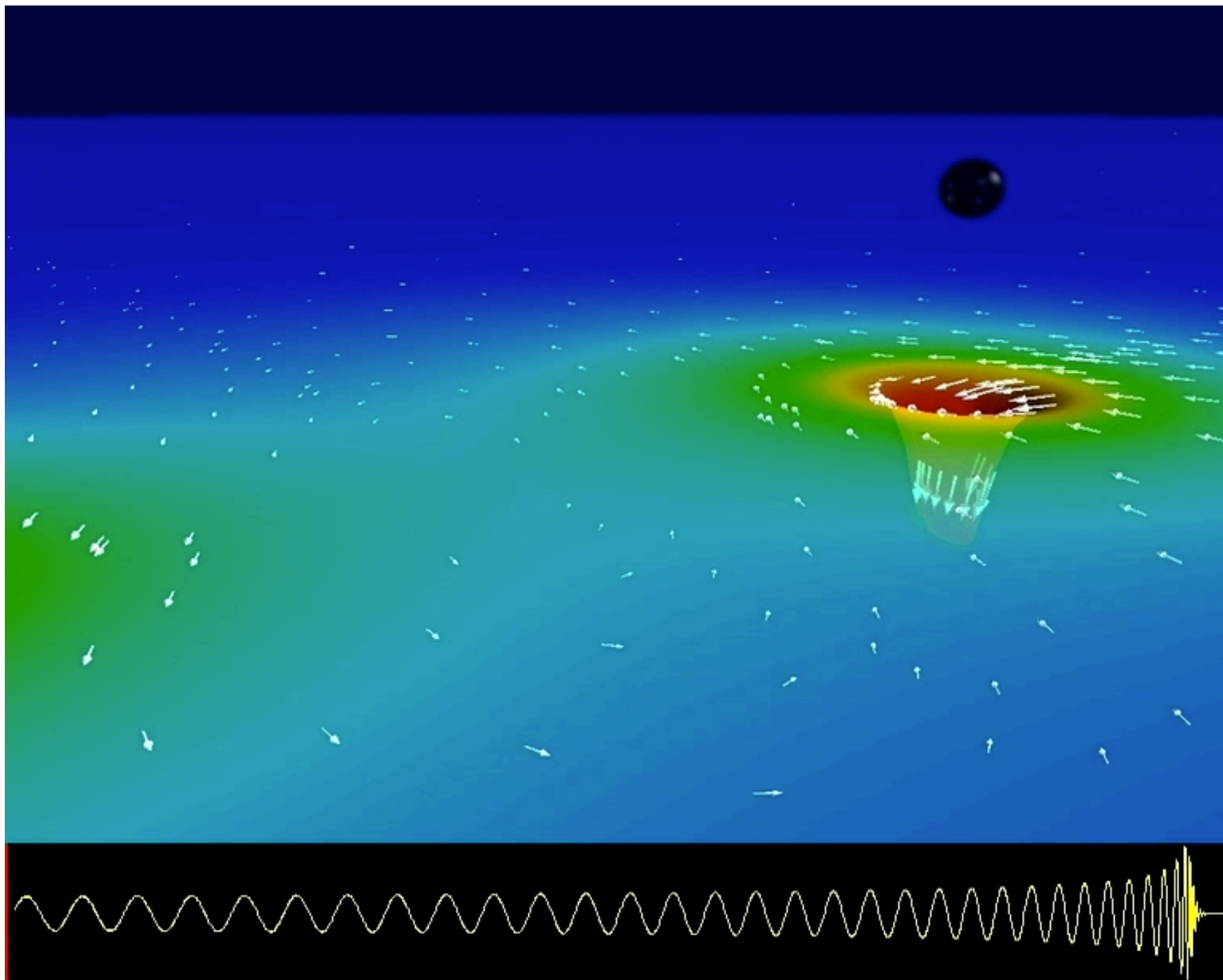
GR vs. Newton:

GR needs stronger pressure gradient for equilibrium.

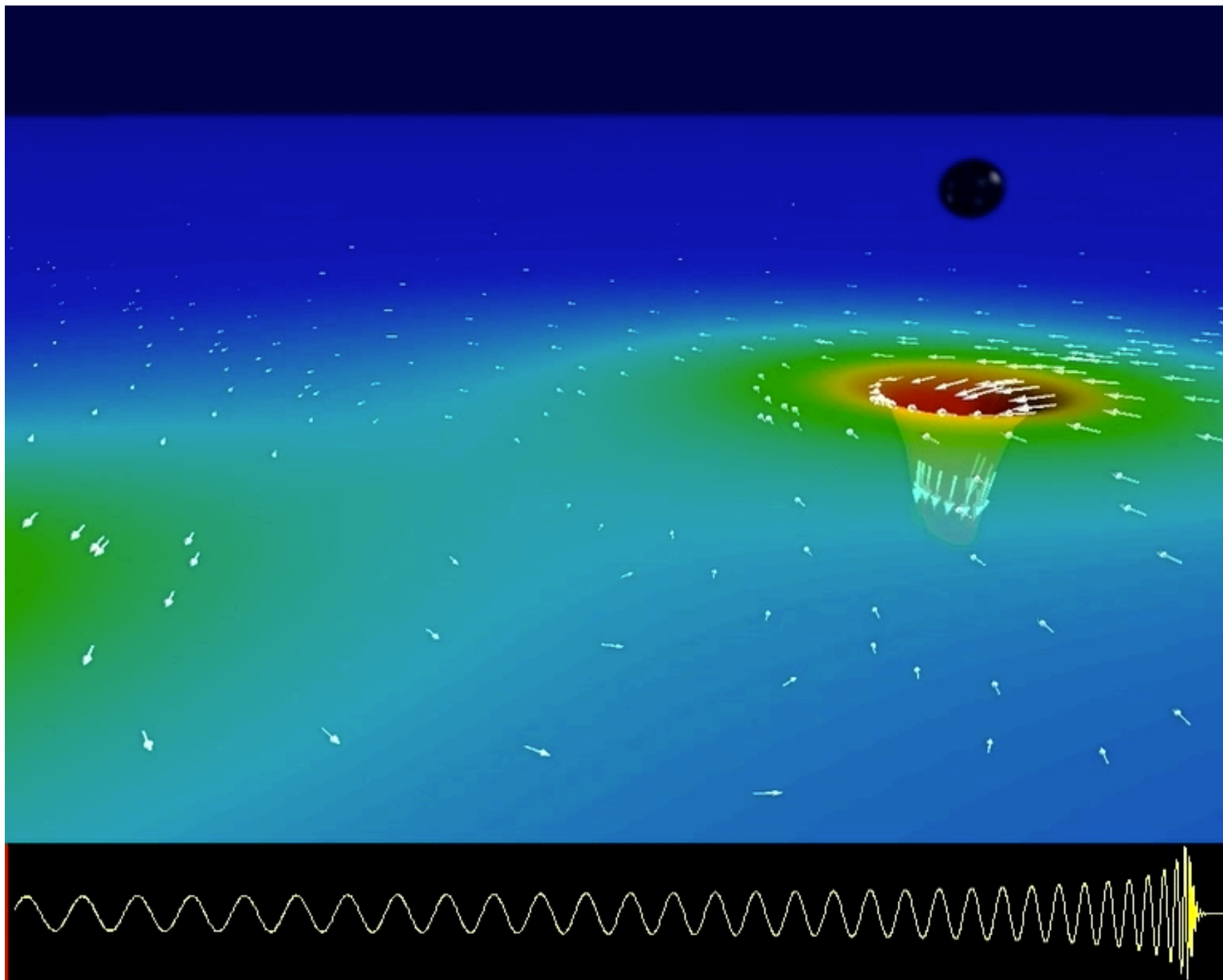
- Qualitative behaviour captured by star with: $\rho=\text{const.}$, P_0 = central pressure.
 - Newton: solution always exists,
 - GR: the central pressure can diverge for finite ρ
 $\Rightarrow$ there is a maximal mass for stability:

$$P(r) = \frac{2}{3} \pi \rho_0^2 (R^2 - r^2)$$

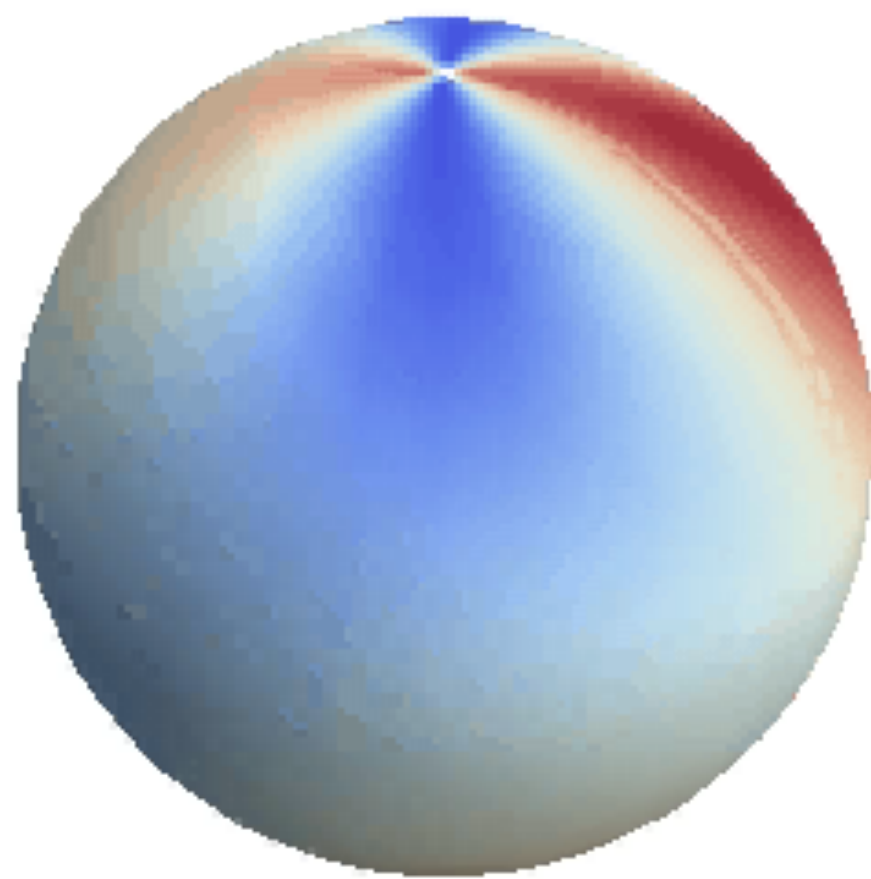
$$M_{max} = \frac{4}{9\sqrt{3\pi\rho_0}}$$

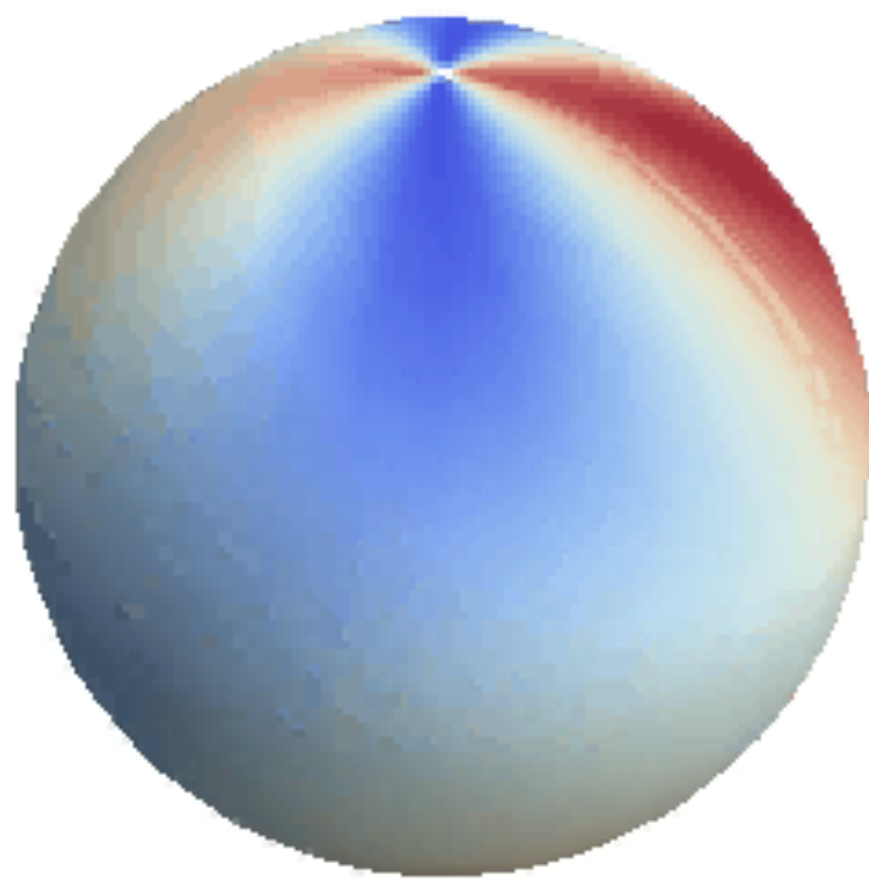


Visualisation:
SXS
collaboration



Visualisation:
SXS
collaboration





Indirect evidence of gravitational radiation: PSR 1913+16

- 1960's: Chandrasekhar - reaction radiation problem (2.5 PN)
- 1963: Quasars associated with black holes
- 1967: Jocelyn Bell and Antony Hewish discover the signal from the first pulsar PSR B1919+21
- BH/NS: mathematical entities-> astrophysical bodies
- 1974: Hulse and Taylor discover binary neutron star

PSR 1913+16 orbit will continue to decay over the next ~300 million years, until coalescence. Gravitational wave emission will be strongest near the end.



J. Taylor



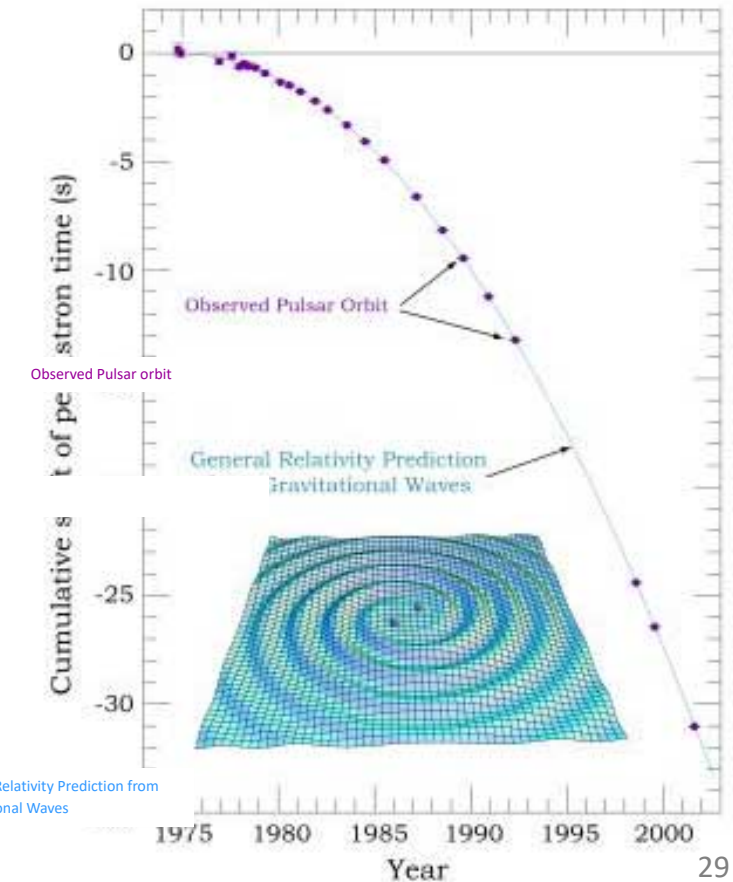
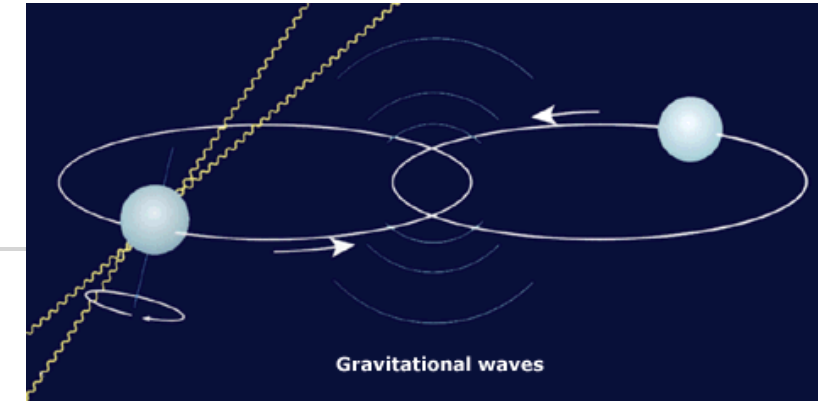
R. Hulse

Nobel Prize in Physics 1993

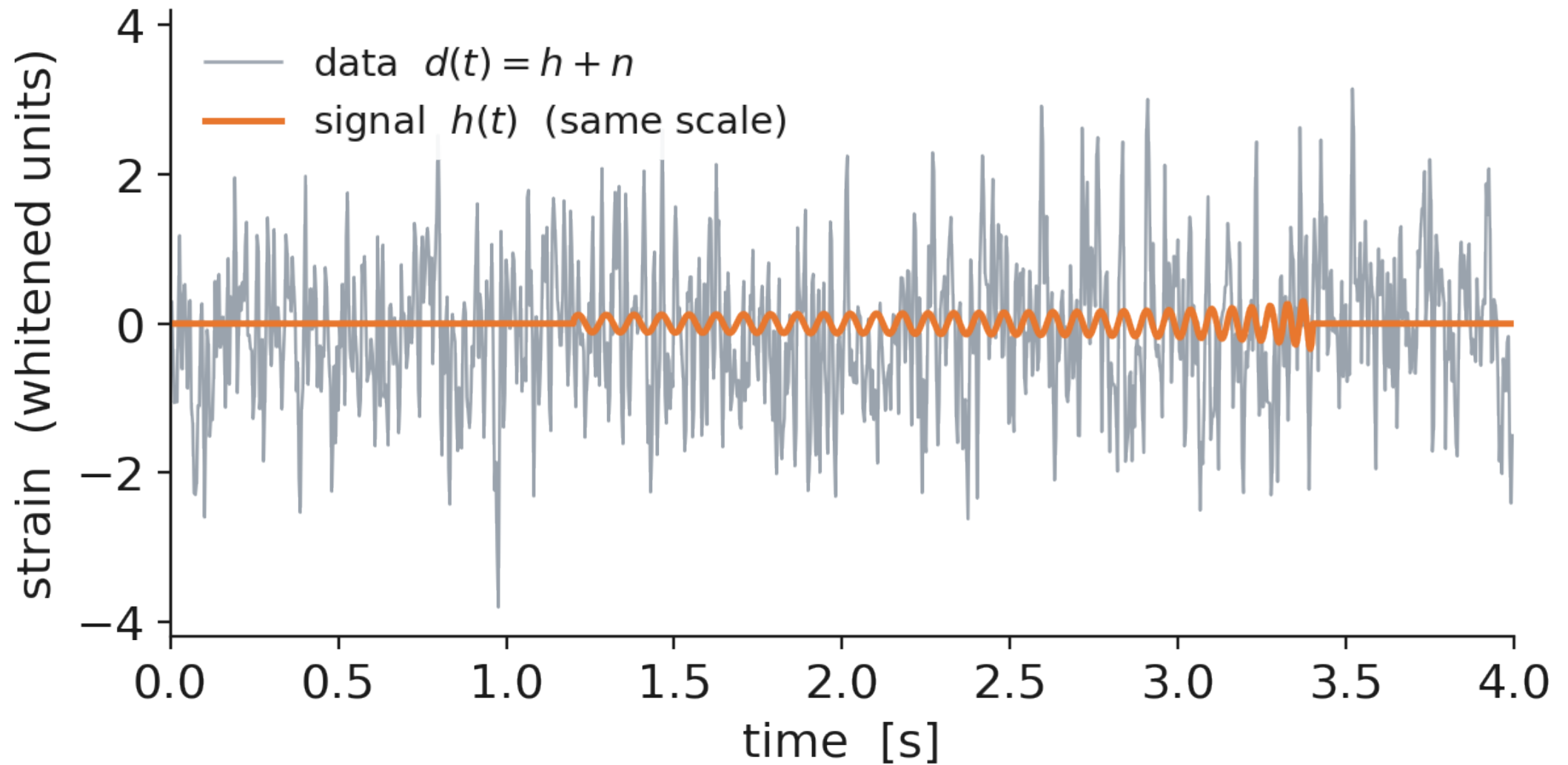


Observed over the course of decades

- Slow orbital decay
- Gravitational waves carrying away orbital energy



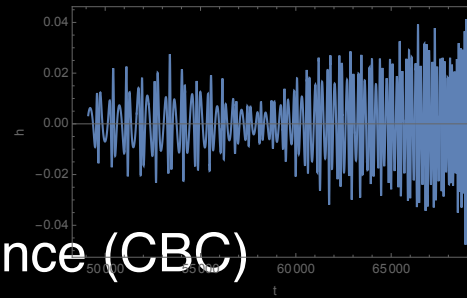
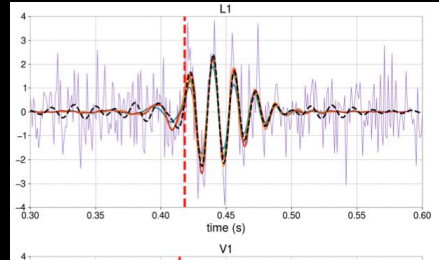
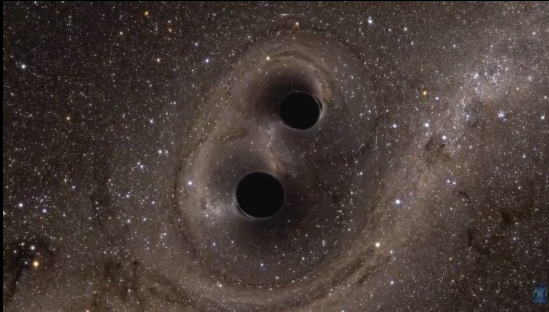
Data analysis



Types of Gravitational Wave Signals & Sources

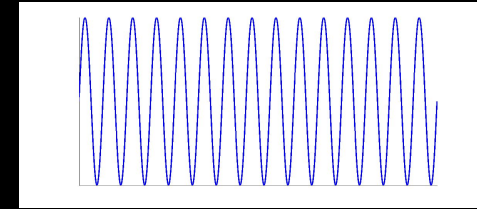
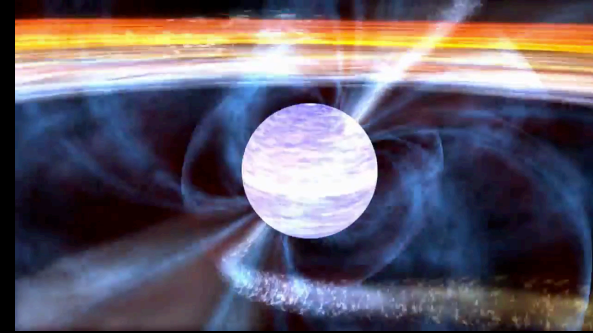
MATCHED FILTER

TRANSIENT



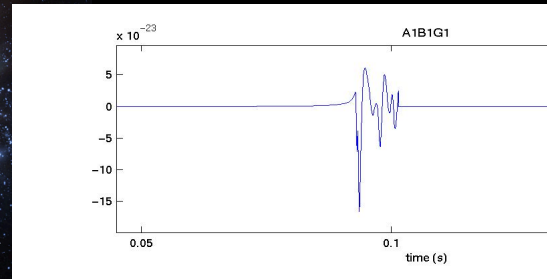
compact binary coalescence (CBC)

PERSISTENT



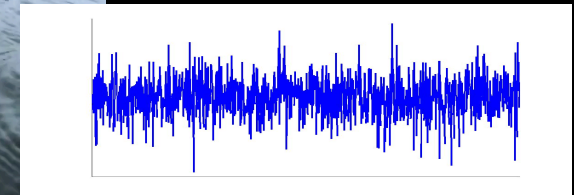
Asymmetric neutron stars

UNMODELED



BURSTS

core collapse supernovae, the unknown



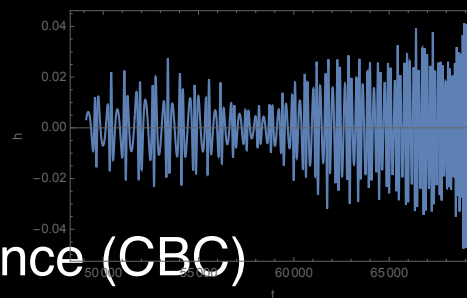
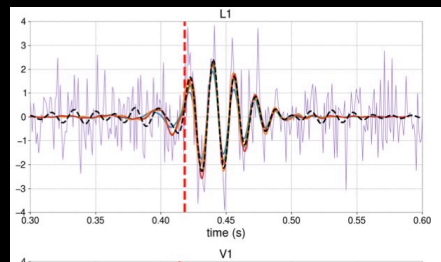
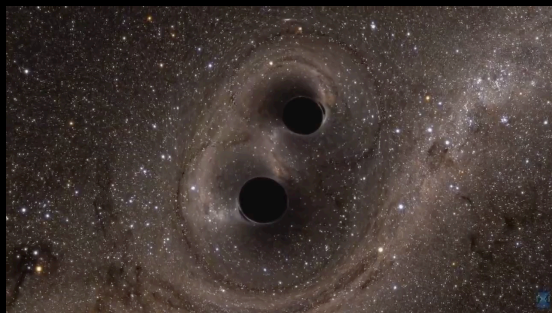
STOCHASTIC BACKGROUND

Cosmological origins, superposition of unresolved signals

Types of Gravitational Wave Signals & Sources

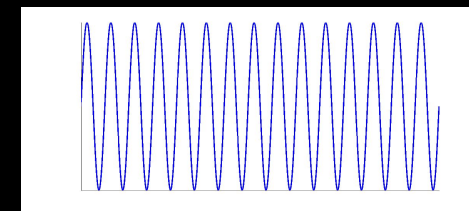
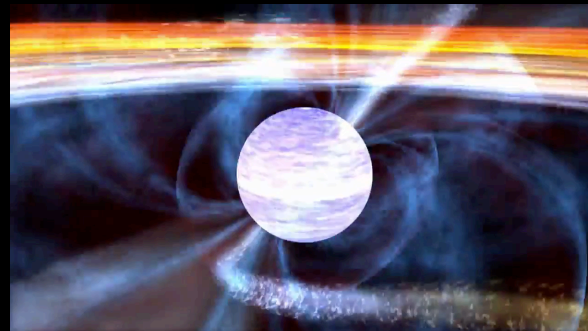
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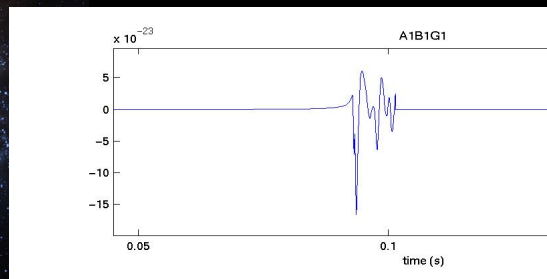
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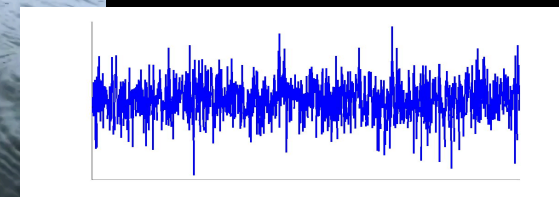
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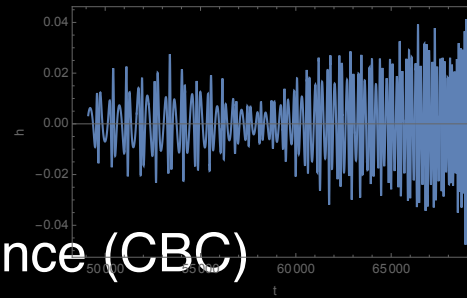
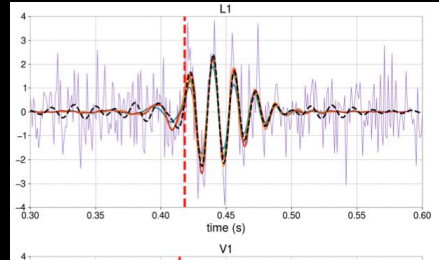
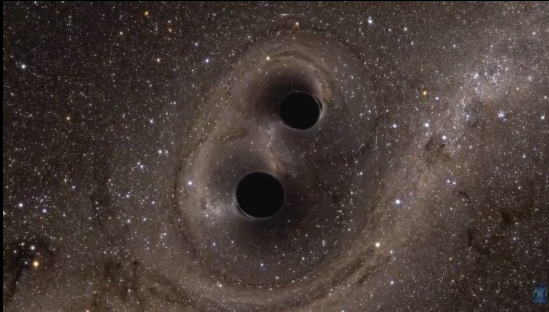
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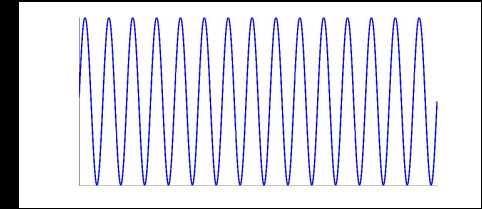
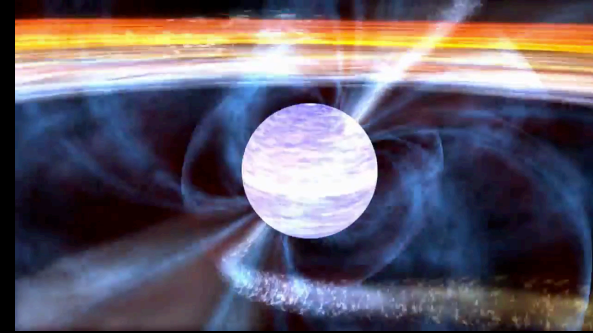
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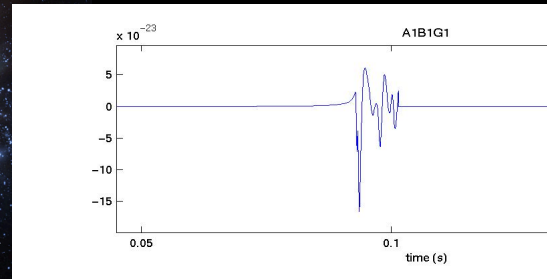
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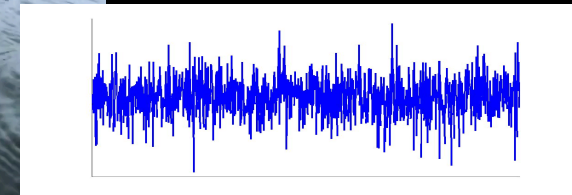
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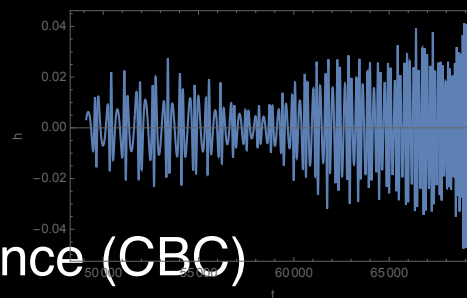
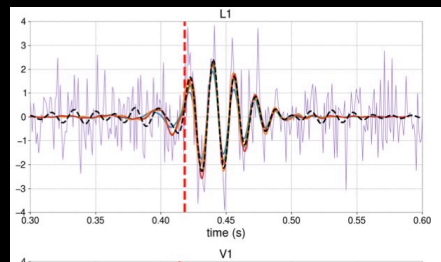
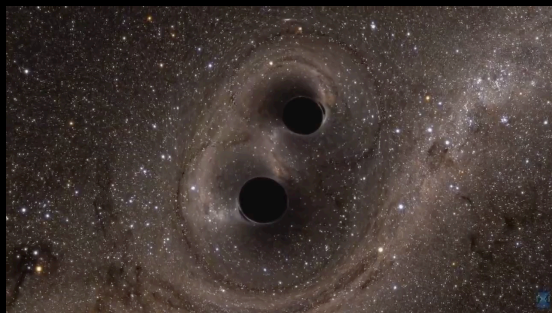
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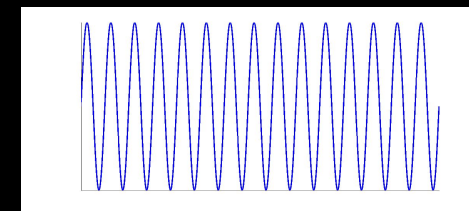
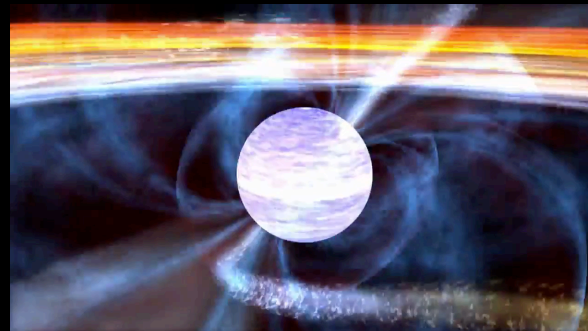
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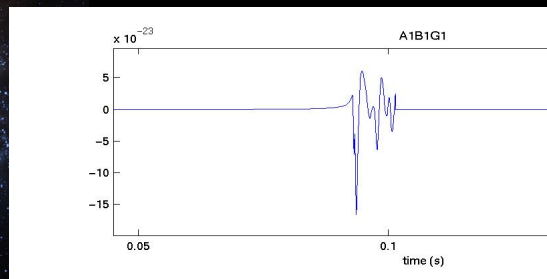
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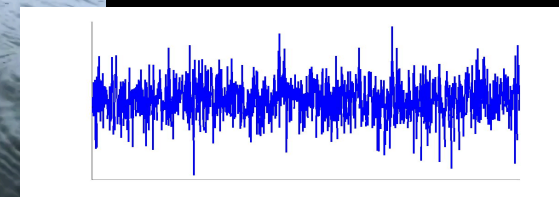
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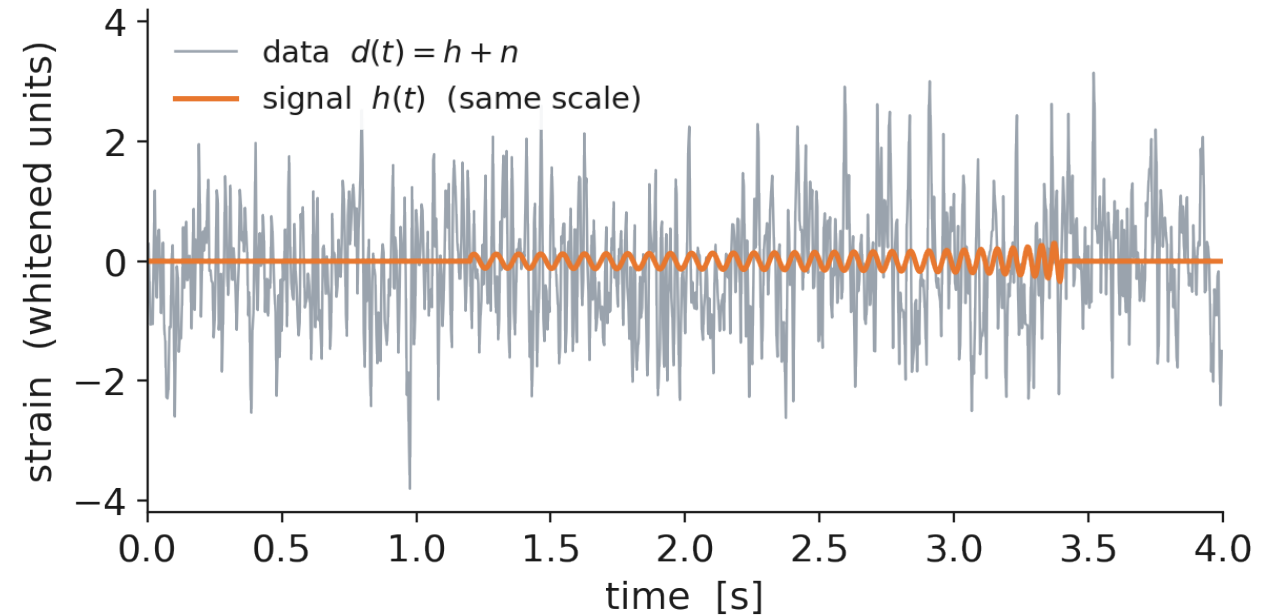
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Noise & Fourier conventions

- **Idealized noise** — stationary and Gaussian with zero mean; real data add **lines and glitches**, but the Gaussian theory sets the framework.
- We can **characterise the Gaussian noise** by the **Power spectral density (PSD)** or **Amplitude Spectral density (ASD)**.
- **Noise is naturally characterised in the frequency domain - use consistent conventions!**
- **One-sided PSD or ASD** - choose $f > 0$ ($f=0$ is typically not accessible).

**DATA
MODEL**

$$d(t) = h(t; \theta) + n(t)$$



FOURIER CONVENTION

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt, \quad \tilde{h}(-f) = \tilde{h}^*(f)$$

DEFINITION OF THE ONE-SIDED PSD

$$\langle \tilde{n}(f) \tilde{n}^*(f') \rangle = \frac{1}{2} S_n(f) \delta(f - f'), \quad f \geq 0 \text{ (one-sided)}$$

Noise & Fourier conventions

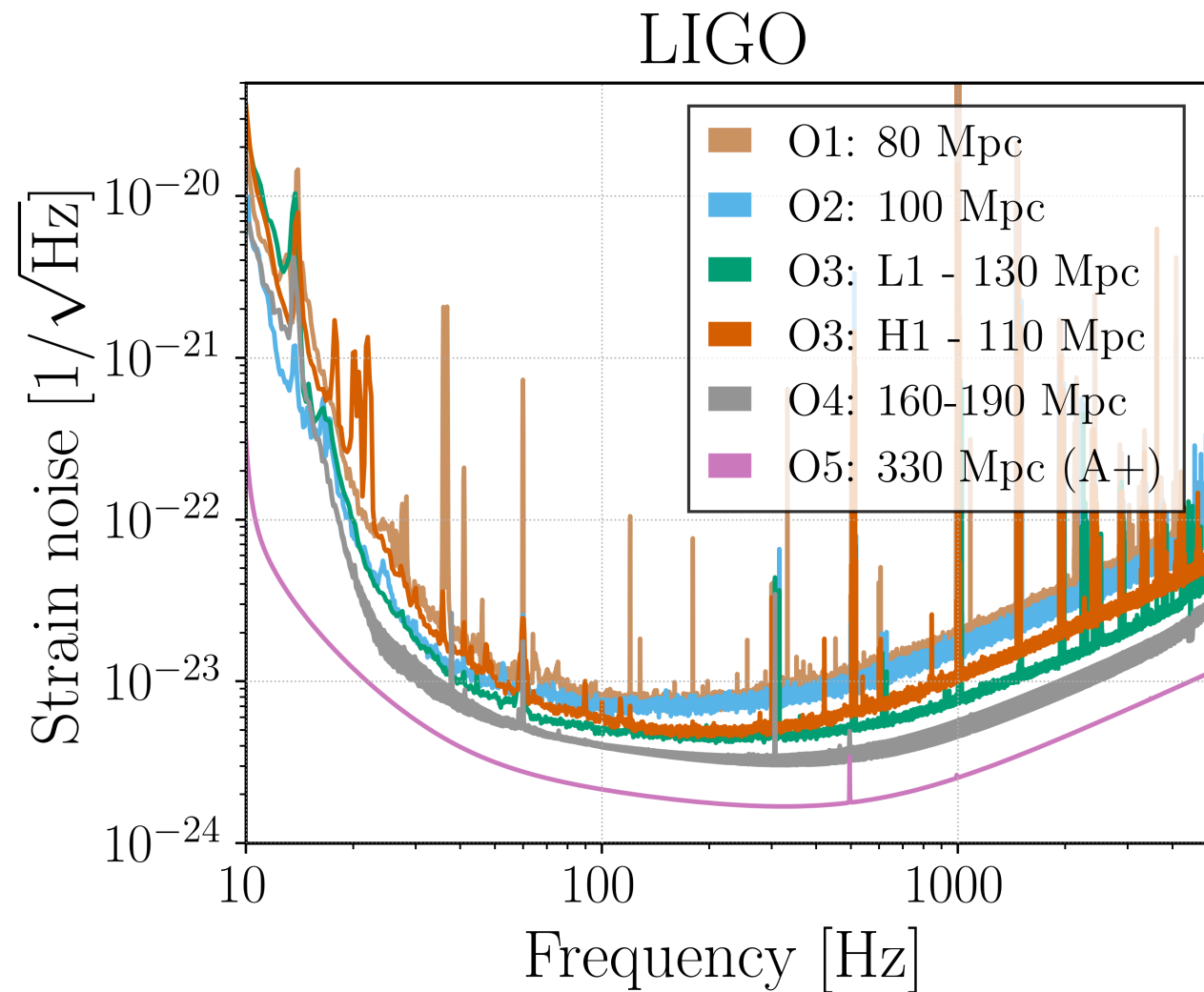
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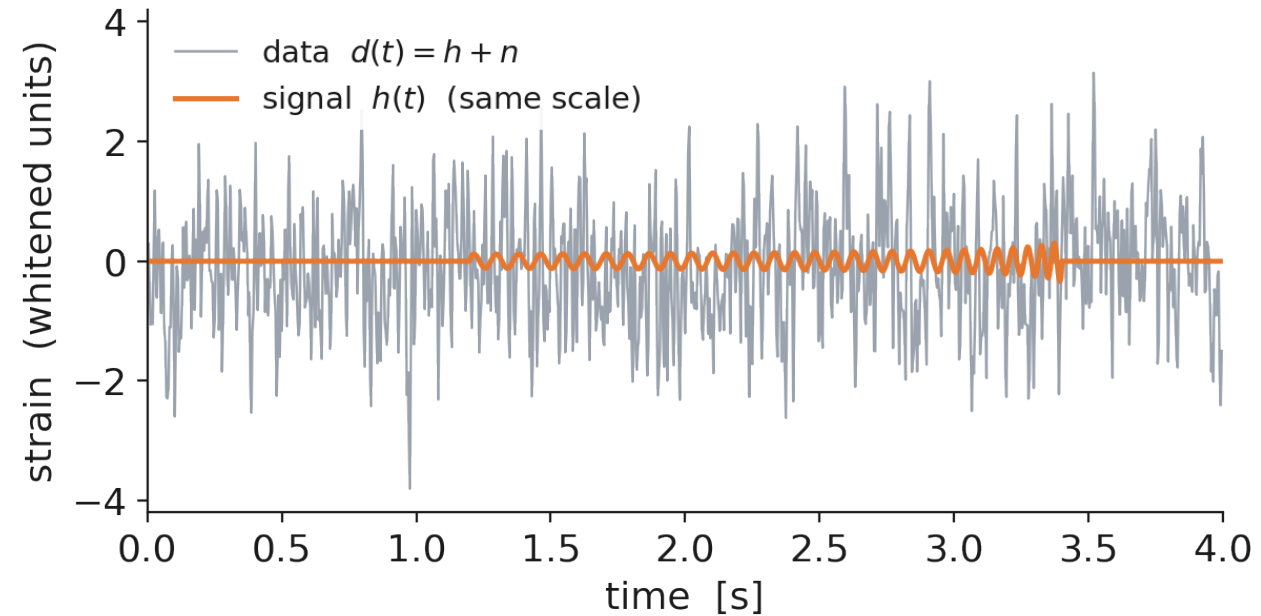


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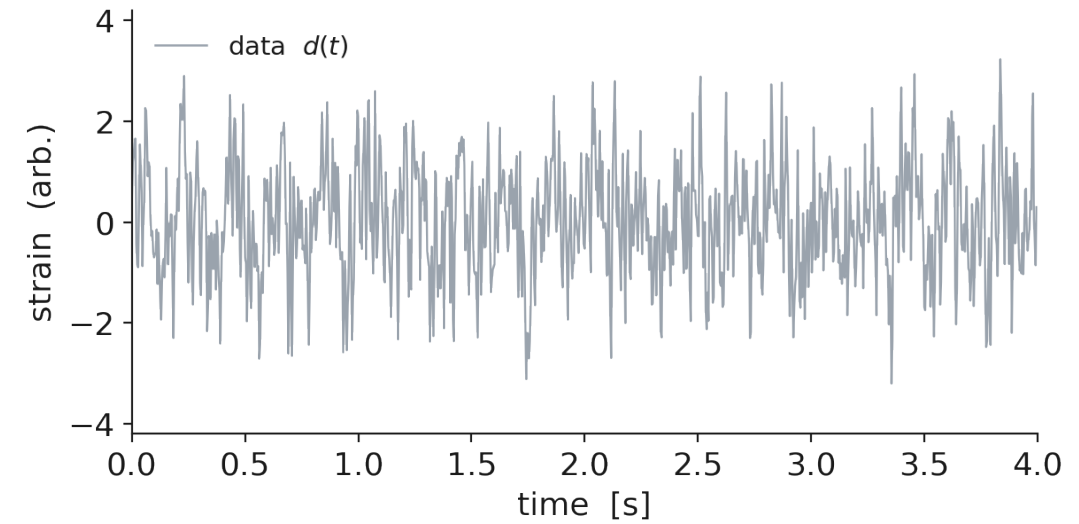
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How to find signals in noise - basic idea

SIMPLE EXAMPLE: MONOCHROMATIC SIGNAL

$$h(t, \omega) = e^{i\omega t}$$



How to find signals in noise - basic idea

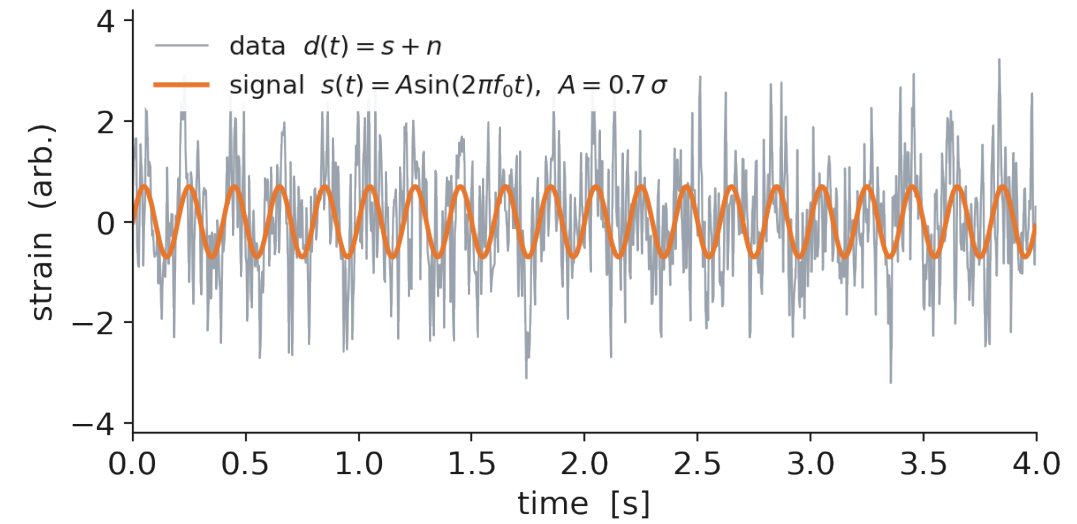
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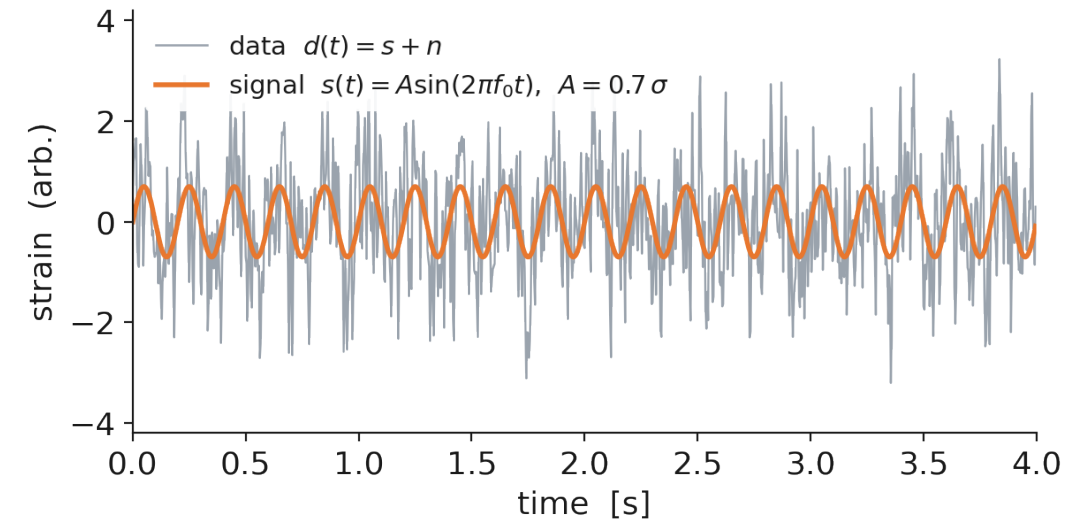
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SIMPLE EXAMPLE: MONOCHROMATIC SIGNAL

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Idea: Fourier transform

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How to find signals in noise - basic idea

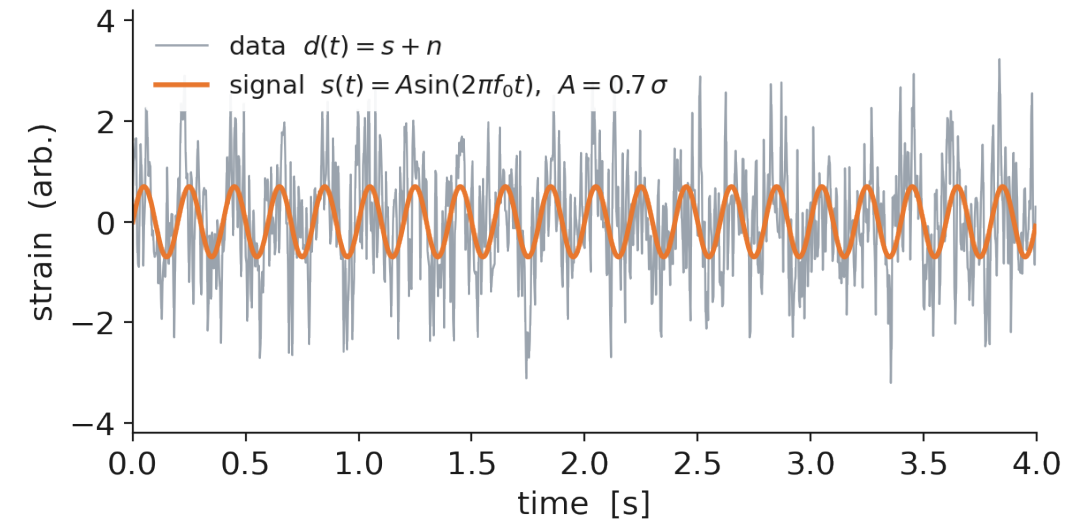
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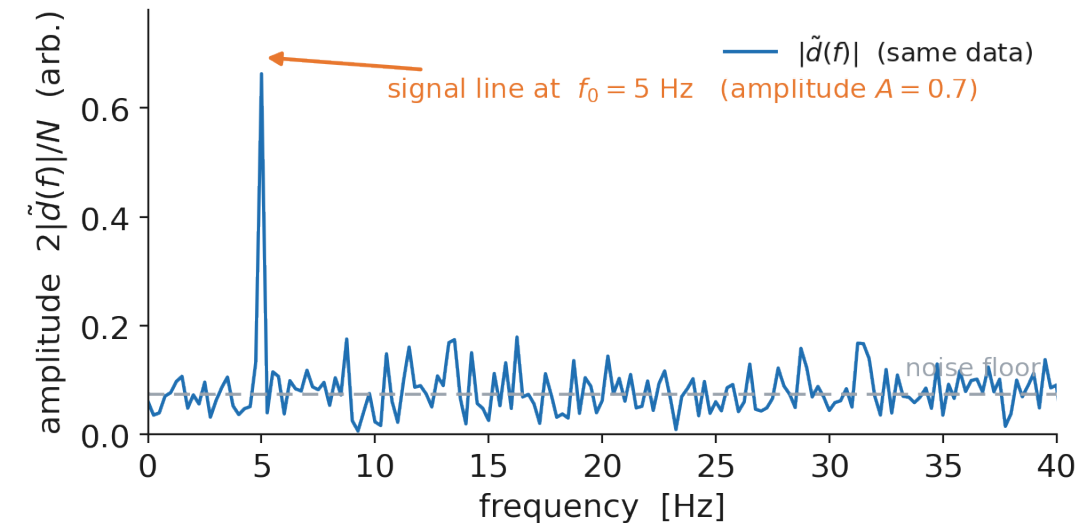
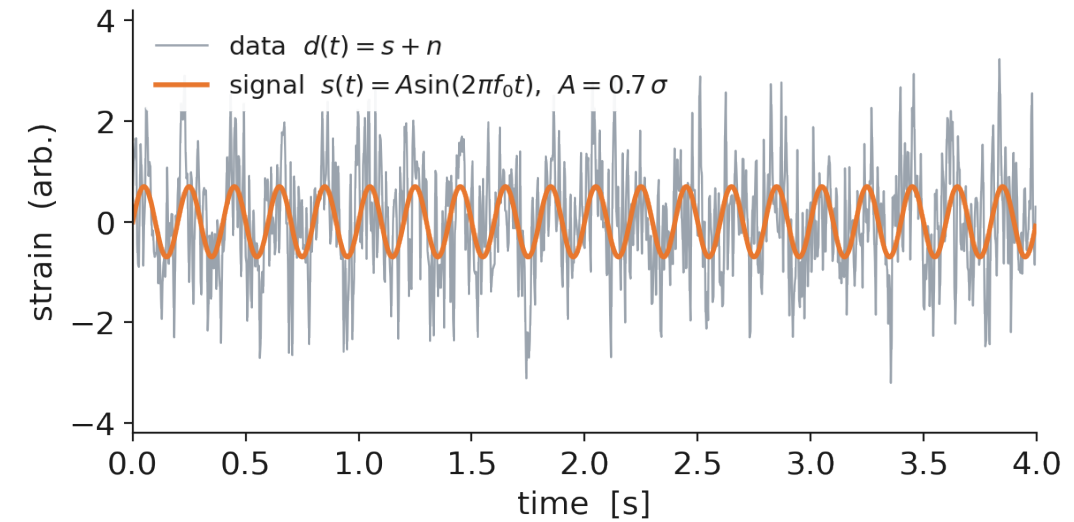
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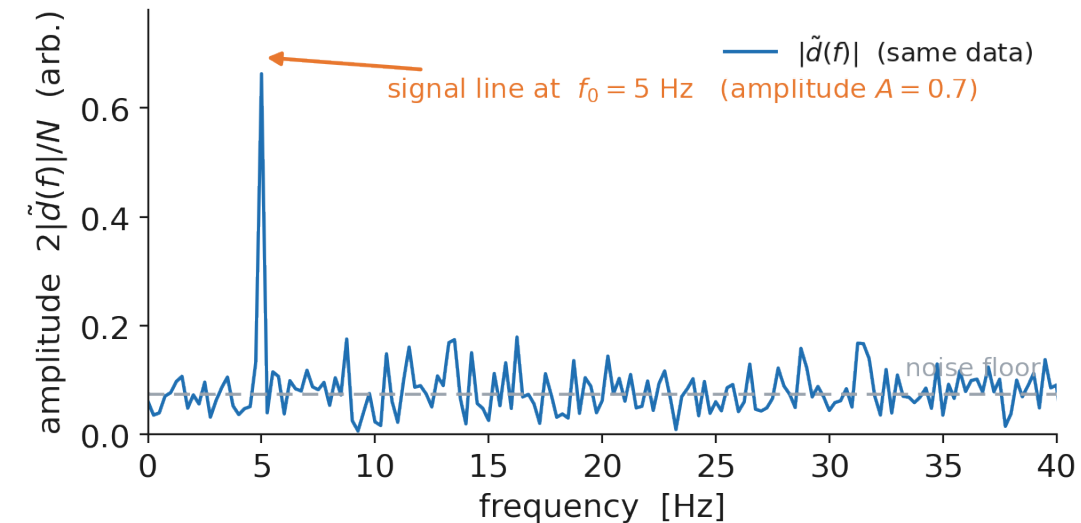
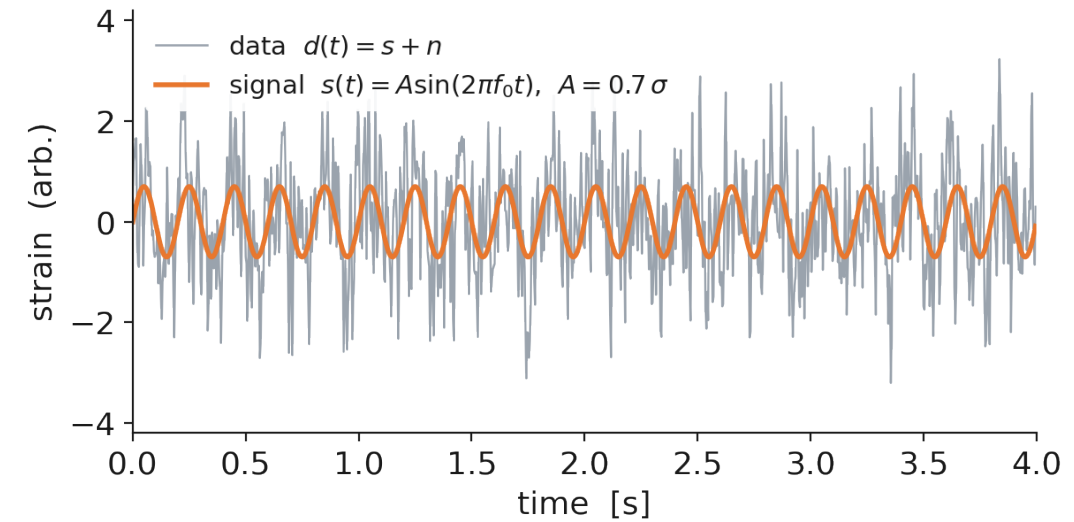
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Generalize to

$$\sim \int_{-\infty}^{\infty} \text{Template}(t) \text{Data}^*(t) dt$$



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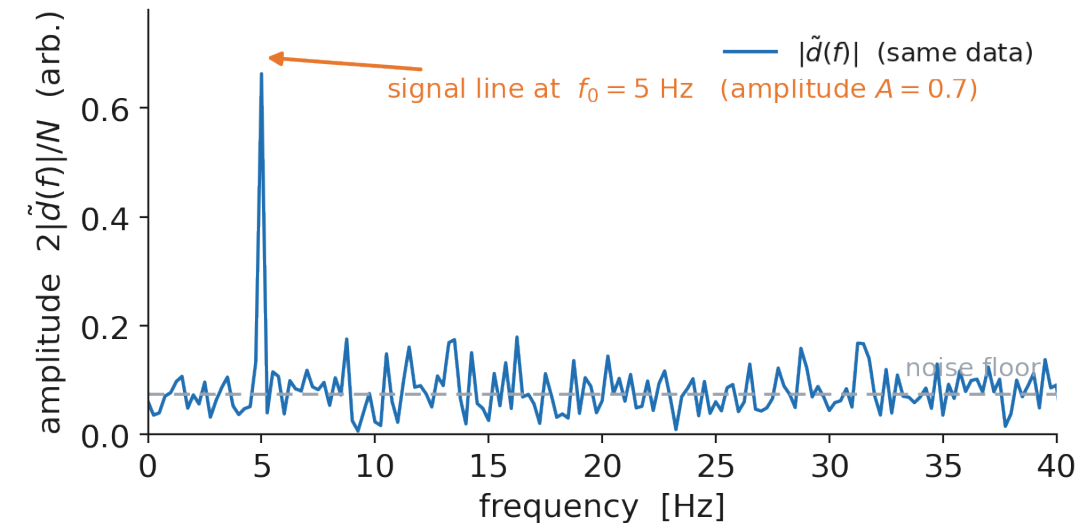
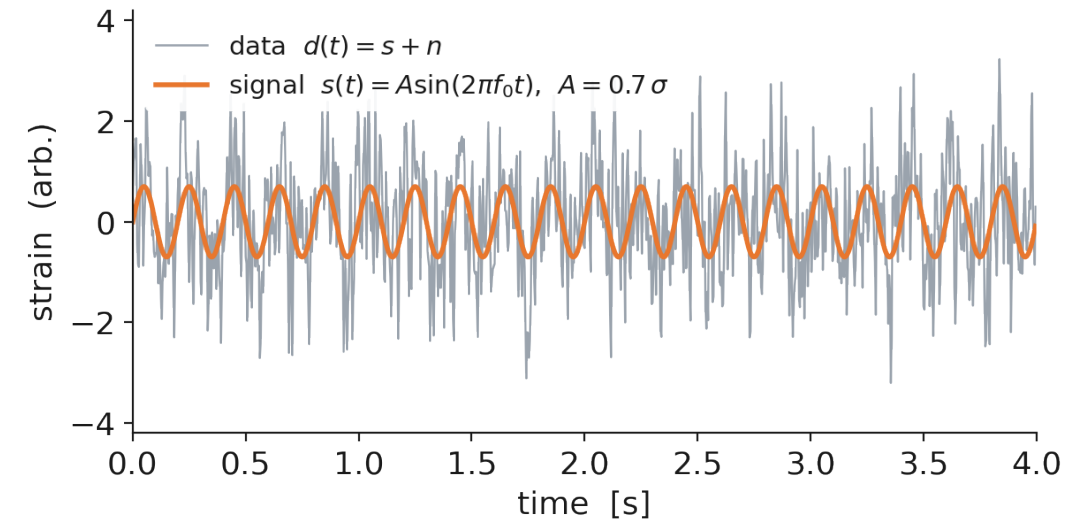
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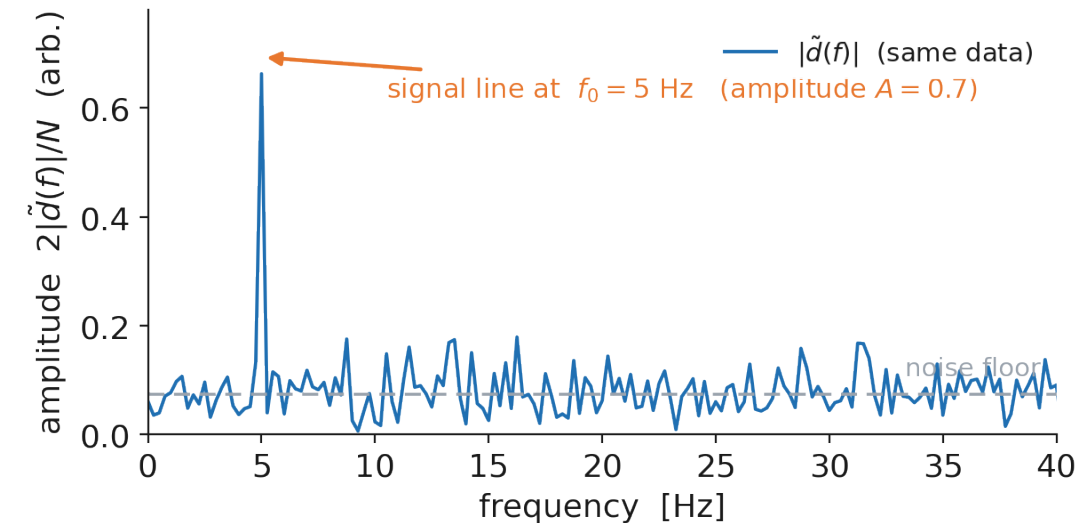
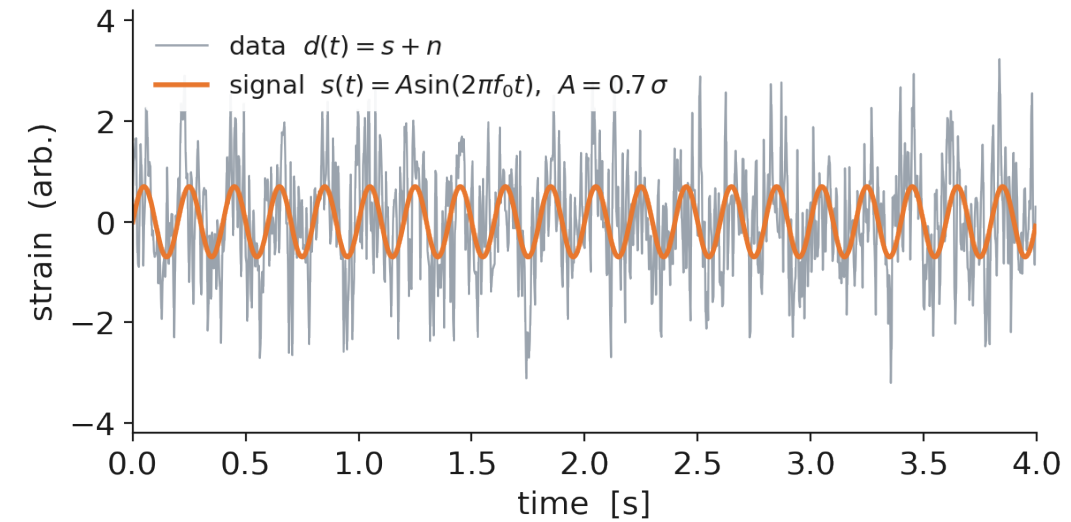
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=> This is a noise-weighted scalar product => matched filtering.

Data analysis methods (for transient GWs)

- **Modeled signals:** matched filter - optimal analysis using accurate waveform models as signal templates.

BASED ON NOISE-WEIGHTED SCALAR PRODUCT

$$(a | b) = 4 \Re \int_0^\infty \frac{\tilde{a}(f) \tilde{b}^*(f)}{S_n(f)} df$$

WHITENED FORM

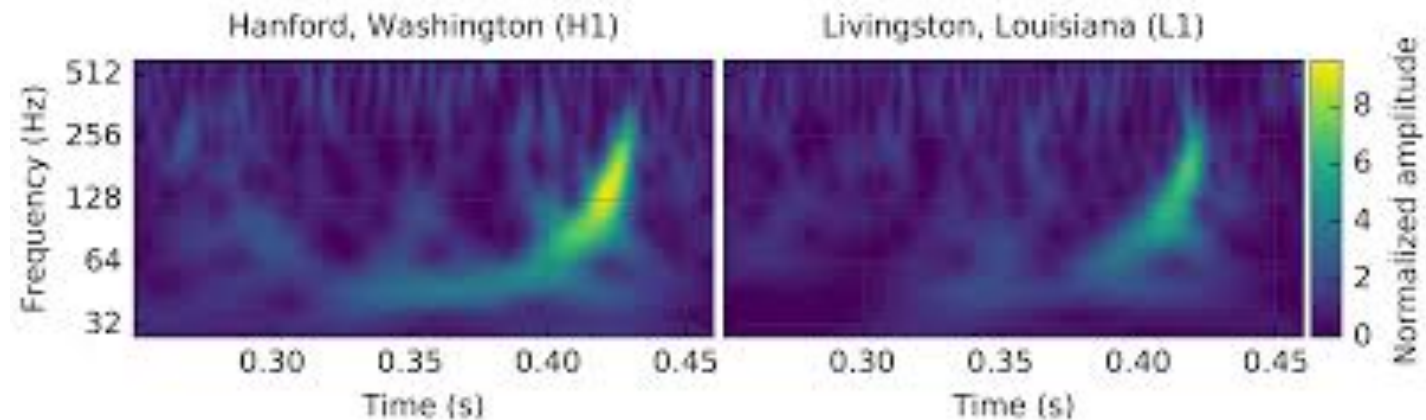
$$\tilde{a}_w(f) \equiv \sqrt{\frac{2}{S_n(f)}} \tilde{a}(f) \quad \Rightarrow \quad (a | b) = \int a_w(t) b_w(t) dt$$

- **Un-modeled searches:**

Time-frequency pattern recognition.
Weakly tuned to expected signals.

- Identification of sources is limited by detector sensitivity + accuracy of waveforms.

=> want accurate waveform models across astrophysically plausible parameter space.



Searches & Bayesian parameter estimation

- **2-step LIGO/Virgo/KAGRA workflow:**

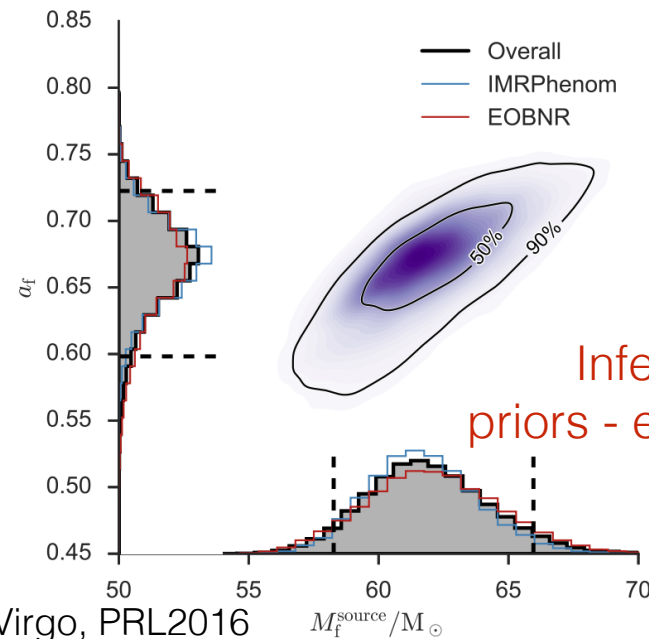
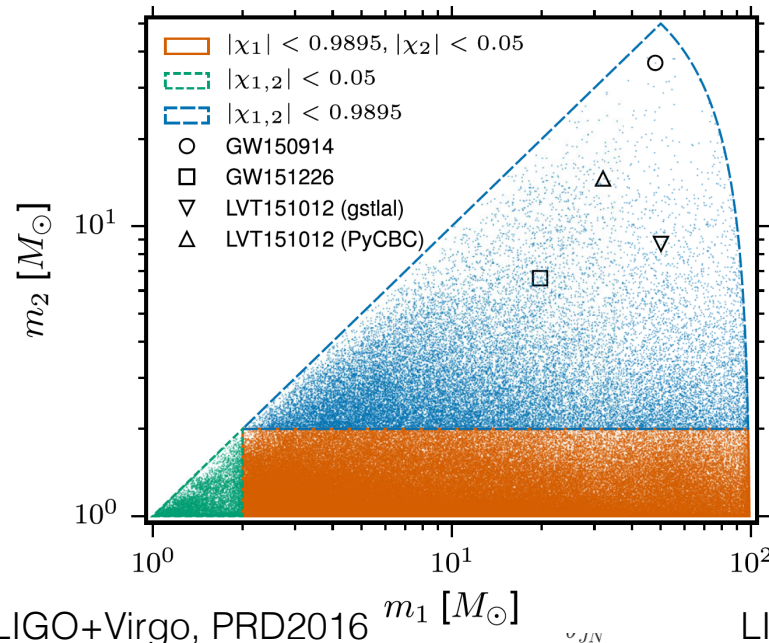
- **Searches in noise dominated data -> detection:**

what is the statistical evidence of seeing a signal above background, fixed template bank or unmodeled searches.

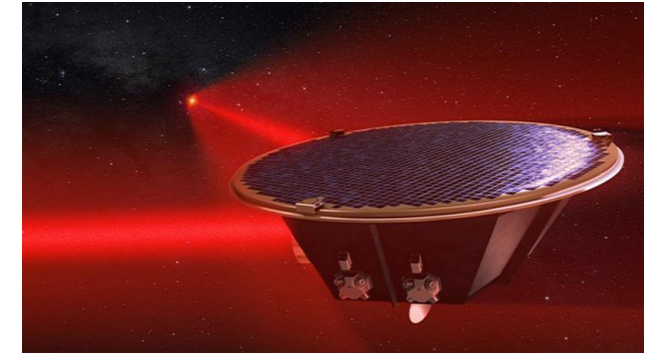
- Test pipeline sensitivity with waveform injections -> astrophysical rates

- **Bayesian parameter estimation for signals that can be modelled, such as CBCs:**

vary templates with random walks in parameter space, using MCMC ...



Inference results depend on priors - especially for broad posteriors!



Data analysis will be more complicated with LISA

- Many signals overlap - data are signal dominated!
- Need to perform a global fit for many signals + noise at the same time

Bayesian parameter estimation

- Infer source properties (masses, spins, ...) as probability distributions
- Compare data with waveform models using a likelihood function

LVK standard sampler: Bilby with Dynesty (nested sampling)

More info in GWTC-4.0 Methods ([arxiv:2508.18081](#))

Parameter vector:

15 parameters for QC BBH + tidal parameters for BNS/NSBH + calibration parameters for each detector

Standard Assumptions

- Data = detector noise + astrophysical GW signal
- Noise is coloured Gaussian and stationary
- Signal described by CBC waveform models
- General Relativity is assumed

Likelihood of data being described by model with parameter vector θ

$$p(\theta|d) = \frac{p(d|\theta)\pi(\theta)}{p(d)}$$

Posterior probability distribution

Model evidence

Prior probability distribution

Wide agnostic priors based on preliminary PE

Slide adapted from Lucy Thomas vi Héctor Estellés

Likelihood & the matched-filter SNR

GAUSSIAN LIKELIHOOD

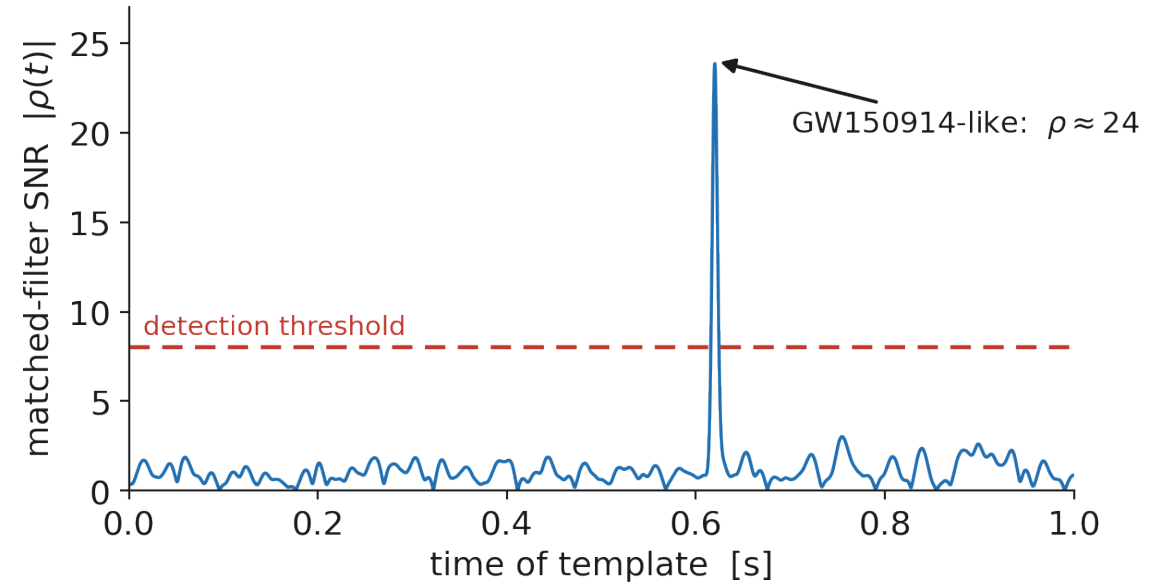
$$p(d | \theta) \propto \exp\left[-\frac{1}{2} (d - h(\theta) | d - h(\theta))\right]$$

$$\ln p(d | \theta) = (d | h) - \frac{1}{2} (h | h) + \text{const}$$

- Maximising the likelihood = minimising the residual power $(r | r)$, $r = d - h$.
- SNR ρ^2 is the power removed from the data.
- Typical SNR required for detection: SNR ~ 8 .
How to measure that?

SNR, OPTIMAL SNR & THE RESIDUAL

$$\rho = \frac{(d | h)}{\sqrt{(h | h)}}, \quad \rho_{\text{opt}}^2 = (h | h) = 4 \int_0^\infty \frac{|\tilde{h}(f)|^2}{S_n(f)} df \quad (r | r)_{\text{min}} = (d | d) - \rho^2$$



Sliding the template over the data: a signal stands out as a sharp SNR peak.

Likelihood & the matched-filter SNR

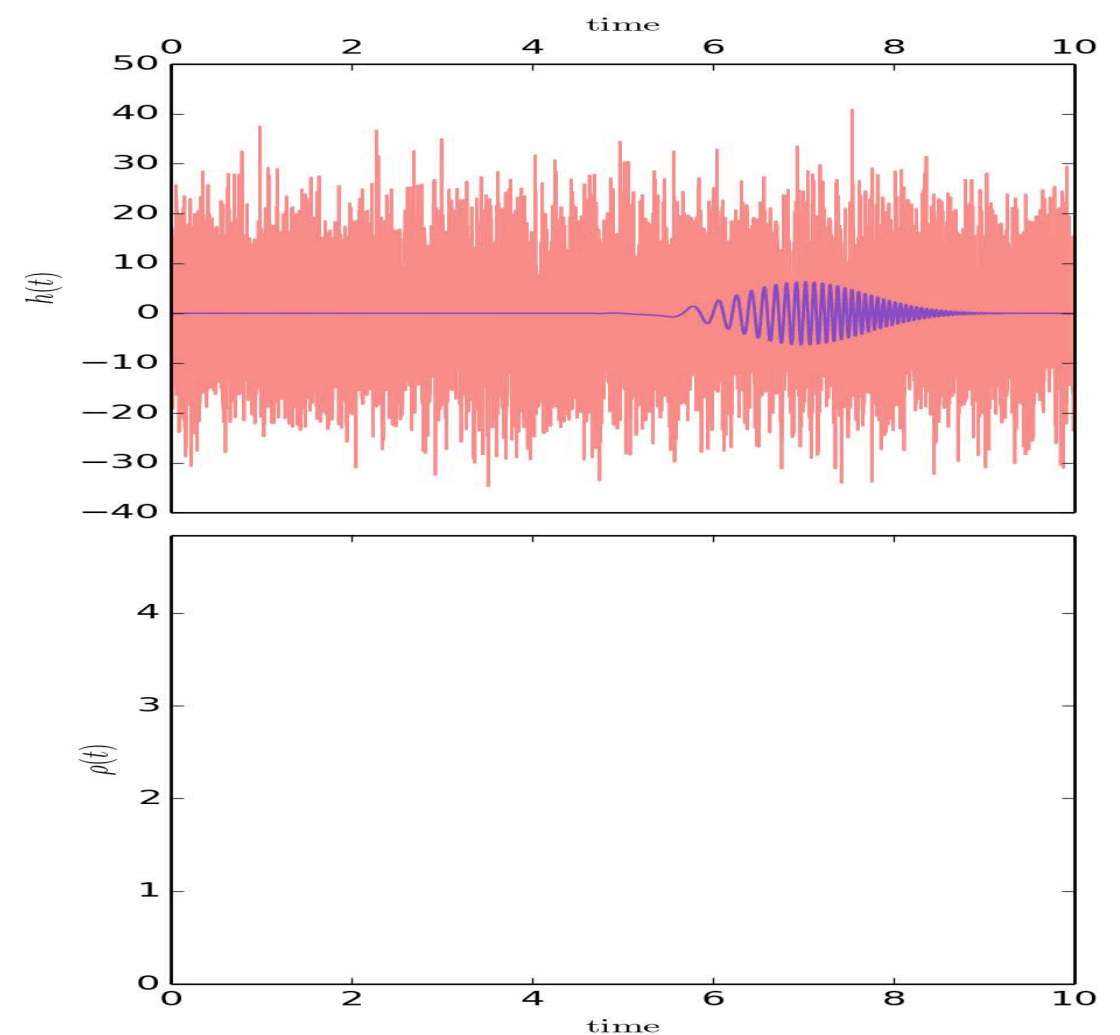
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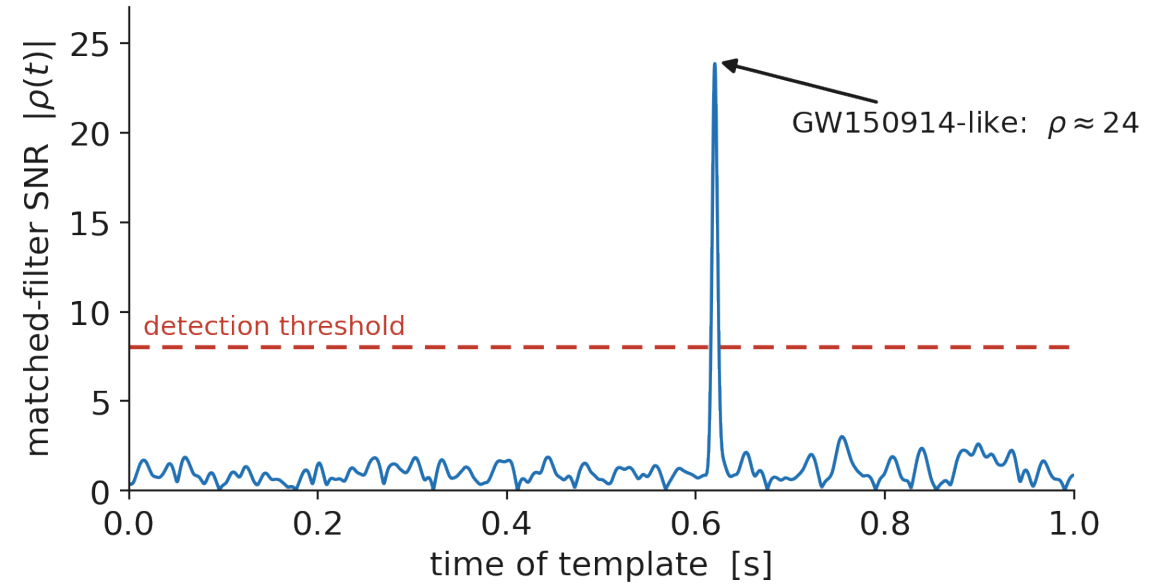
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Sliding the template over the data: a signal stands out as a sharp SNR peak.

Priors: flat where possible — but flat in what?

- **Posterior = likelihood × prior** — with weak data or strong degeneracies the prior is important!
- **Uninformative by symmetry** — uniform in coalescence phase, polarisation angle and arrival time; isotropic on the sky and in orientation (uniform in $\cos \iota$).
- **Masses: 'flat' is ambiguous** — flat in (m_1, m_2) is not flat in $(\mathcal{M}, q)$: samplers explore $(\mathcal{M}, q)$ and correct with the Jacobian.
- **Distance needs a cosmology** — 'uniform in space' means D^2 nearby; cosmologically, uniform in comoving volume with H_0, Ω_m assumed — and source-frame masses inherit that cosmology through the redshift.
- **Spins & populations** — isotropic directions and uniform magnitudes by default; population analyses swap in astrophysical priors.

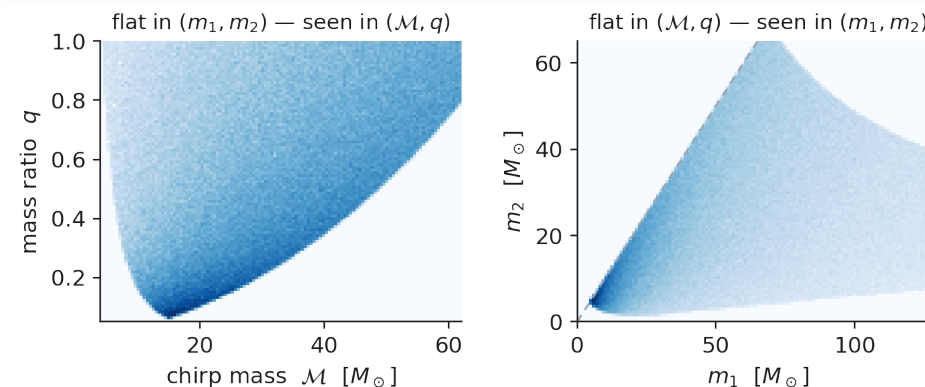
THE EXACT MASS-PRIOR JACOBIAN

$$\text{flat in } (m_1, m_2) \iff \pi(\mathcal{M}, q) \propto \mathcal{M} \left[\frac{1+q}{q^3} \right]^{2/5}$$

A DISTANCE PRIOR ASSUMES A COSMOLOGY

$$\pi(D_L) \propto D_L^2 \quad (\text{nearby}); \quad \text{cosmological: } \pi(z) \propto \frac{dV_c}{dz} \Rightarrow (H_0, \Omega_m)$$

$$m_{\text{det}} = (1+z) m_{\text{source}} \Rightarrow \text{source-frame masses inherit the cosmology}$$



Same samples, two parametrizations: 'flat' in one is curved in the other.

KL & JS divergences: quantify difference in distributions p, q

- **Kullback–Leibler:** information gain: p is the distribution of interest, q the reference, e.g. the prior.
- **Jensen–Shannon:** symmetries KL - for quantifying the deviation between 2 posteriors.
 - Bounded by 1 bit.

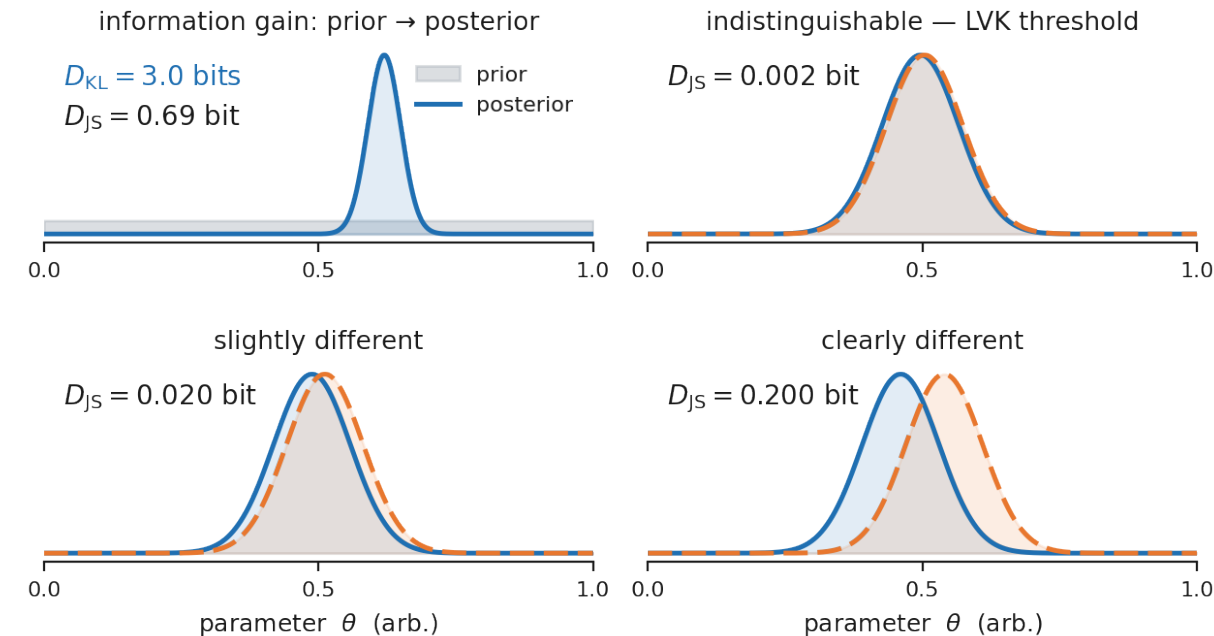
UNITS

nats: use $\ln$; bits: use $\log_2$; 1 bit = $\ln 2 \approx 0.693$ nat

DIVERGENCES

$$D_{\text{KL}}(p \parallel q) = \int p(\theta) \ln \frac{p(\theta)}{q(\theta)} d\theta \geq 0 \quad (\text{asymmetric; } / \ln 2 \rightarrow \text{bits})$$

$$D_{\text{JS}}(p \parallel q) = \frac{1}{2} D_{\text{KL}}(p \parallel m) + \frac{1}{2} D_{\text{KL}}(q \parallel m), \quad m = \frac{p+q}{2}$$



All values computed from the plotted densities. Panel 1 shows the contrast: JS is capped at 1 bit — the prior $\rightarrow$ posterior pair already uses much of that bound — while the unbounded KL keeps counting the bits gained.

Comparing waveforms: match & mismatch

- **Match:** the overlap of normalised waveforms, maximised over arrival time and phase — the extrinsic parameters every search maximises automatically.
- **Mismatch:** $MM = 1 - M$ is the fractional SNR lost when filtering with an inaccurate waveform.
- **Fitting factor** — the best match over the whole model manifold: how well the model can imitate the signal at all.

FITTING FACTOR & DETECTION VOLUME

$$FF = \max_{\theta} \mathcal{M}(h_s, h(\theta)), \quad V_{\text{det}} \propto FF^3$$

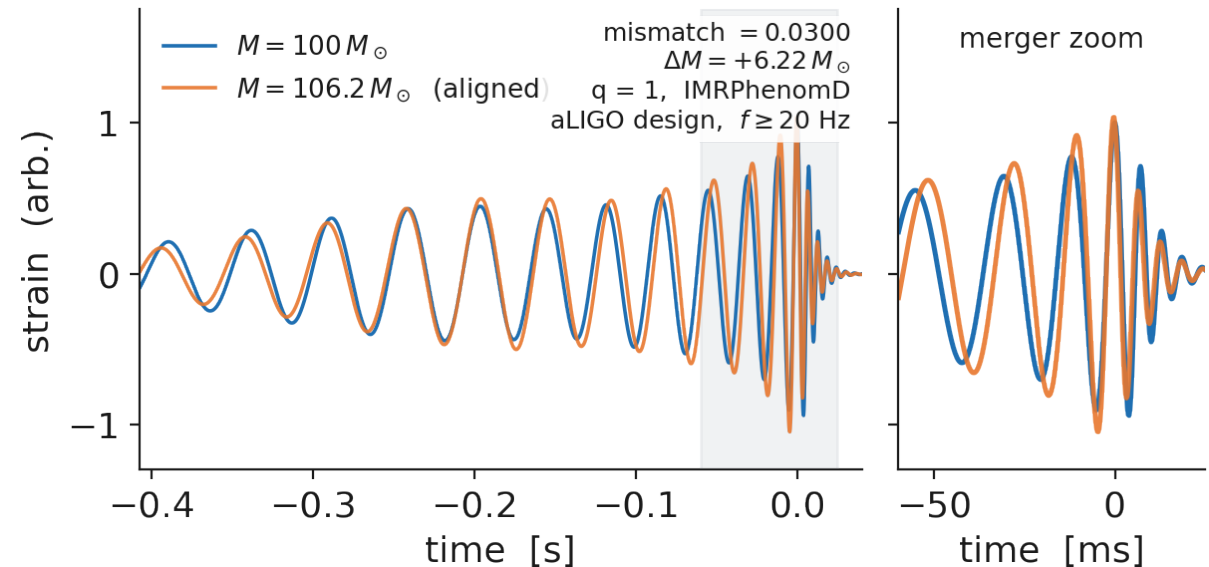
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MATCH — MAXIMISED NORMALISED OVERLAP

$$\mathcal{M}(h_1, h_2) = \max_{t_c, \phi_c} (\hat{h}_1 | \hat{h}_2 e^{i(2\pi f t_c + \phi_c)}), \quad \hat{h} \equiv \frac{h}{\sqrt{(h|h)}}$$

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$$MM \equiv 1 - \mathcal{M}, \quad \rho_{\text{recovered}} = (1 - MM) \rho_{\text{opt}}$$



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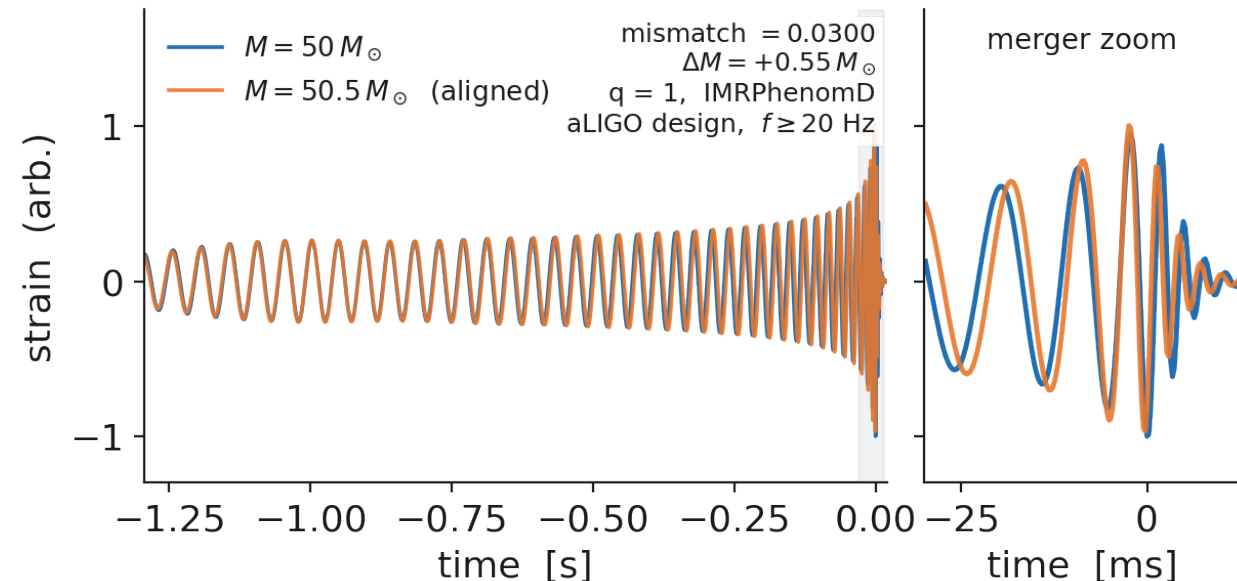
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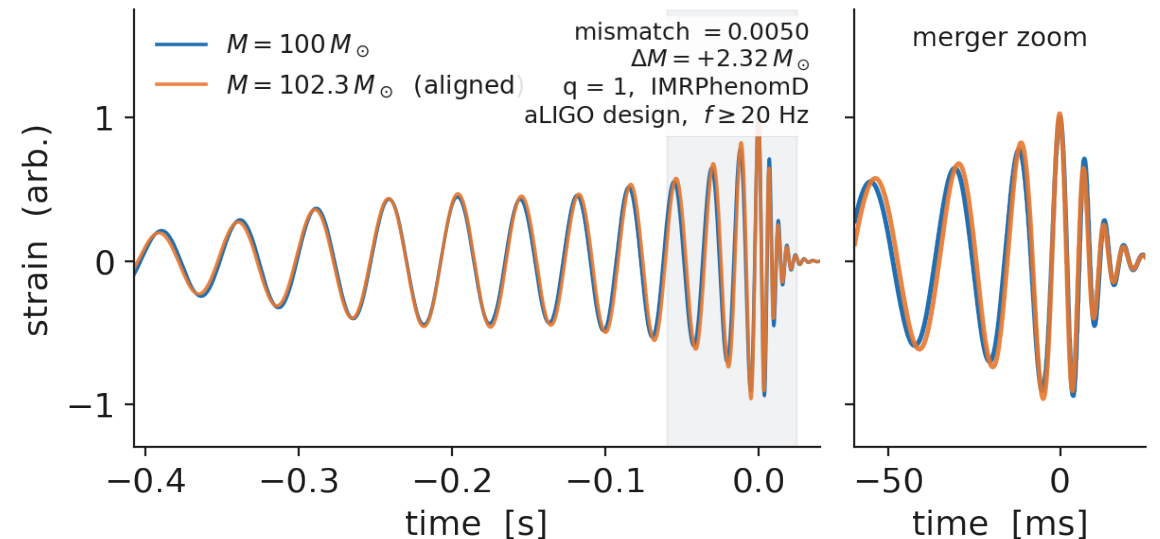
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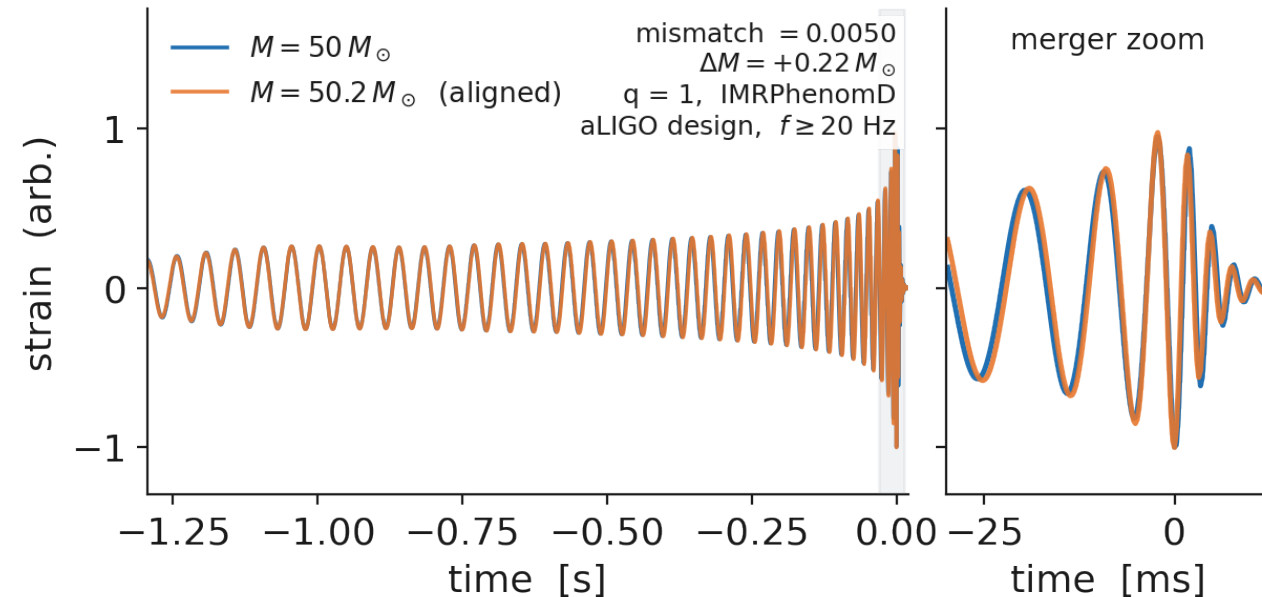
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The waveform space metric

- Inner product => waveform Hilbert-space, noise defines the metric.
- A **waveform model** is a map $\vec{\theta} \mapsto h(\vec{\theta})$ where $\vec{\theta}$ are the n intrinsic and extrinsic parameters: masses, spins, sky location, ...
- The tangent vectors on the n -dimensional **waveform manifold** $\{h(\theta)\}$ are

$$\partial_i h = \frac{\partial h}{\partial \theta^i}$$

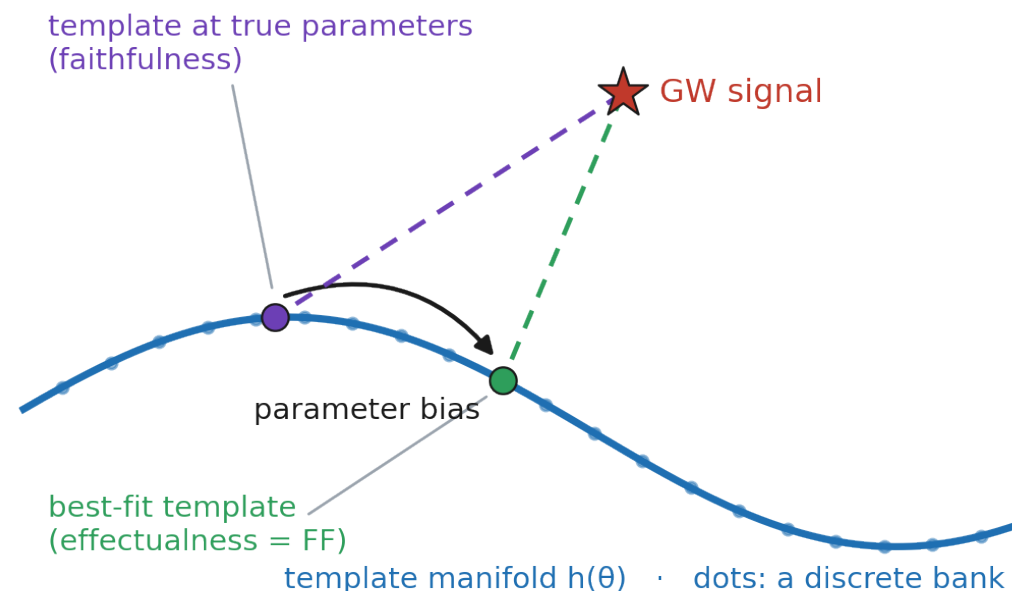
- The inner product then induces a Riemannian metric on the parameter space manifold:

$$g_{ab} = (\partial_a h | \partial_b h)$$

- The match between waveforms with nearby parameters can be expanded as

$$(\hat{h}(\theta) | \hat{h}(\theta + \Delta\theta)) \approx 1 - \frac{1}{2} g_{ij} \Delta\theta^i \Delta\theta^j$$

$$MM(\theta, \theta + \Delta\theta) \approx \frac{1}{2} g_{ij}(\theta) \Delta\theta^i \Delta\theta^j$$



- The metric description is important e.g. to describe the template spacing for detection algorithms.

template space metric = Fisher information matrix

$$g_{ab} = \Gamma_{ab}$$

- Fisher measures how much information a probability model contains about its unknown parameters.

- Remember likelihood:
$$p(d|\theta) \propto \exp \left[-\frac{1}{2} (d - h(\theta) | d - h(\theta)) \right].$$

- **Expand for small parameter changes:**
$$h(\theta + d\theta) \approx h(\theta) + \partial_i h d\theta^i.$$

- One can derive
$$\frac{p(d|\hat{\theta} + d\theta)}{p(d|\hat{\theta})} \approx \exp \left[-\frac{1}{2} \Gamma_{ij} d\theta^i d\theta^j \right]$$

$$p(\theta | d) = \frac{p(d | \theta) p(\theta)}{p(d)}$$

- Compare with Bayes' theorem and Gaussian likelihood (**high SNR**), where Σ is the covariance matrix:
$$p(d\theta) \propto \exp \left[-\frac{1}{2} d\theta^i (\Sigma^{-1})_{ij} d\theta^j \right].$$

- We can estimate the statistical errors (posterior width), and parameter bias due to noise and waveform systematics:

$$\sigma_a = \sqrt{(\Gamma^{-1})_{aa}} \propto \frac{1}{\rho}, \quad \Delta\theta_{\text{noise}}^a = (\Gamma^{-1})^{ab} (\partial_b h | n)$$

CUTLER-VALLISNERI BIAS

$$\Delta\theta_{\text{sys}}^a \simeq (\Gamma^{-1})^{ab} (\partial_b h_{\text{m}} | \delta h), \quad \delta h \equiv h_{\text{true}} - h_{\text{m}}$$
$$\Delta\theta_{\text{sys}}^a \lesssim \sigma_a \quad (\text{waveform accurate enough for inference})$$

Conclusions from Fisher - valid for high SNR, Gaussian likelihood near best θ

- **Problems for non-Gaussian likelihood:**

- Small SNR; degeneracies can deform posterior distributions.

- **Extremely popular:** computationally much cheaper than full Bayesian inference => “cheap” estimates of statistical and systematics errors across parameter space.

Essential for science case studies: What can be measured by future detectors?

- **Popular accuracy (indistinguishability) criterion:**

- This is a crude approximation, avoid if possible!
- Often too restrictive.

- **Statistical errors scale like $1/\text{SNR}$:**

- For higher SNR signals, we need more accurate models => better models as detectors improve!

$$\rho_{\text{res}}^2 \equiv (h_s - h_m \mid h_s - h_m) = 2\rho^2 \text{MM}$$

$$\rho_{\text{res}} \lesssim 1 \quad \Longleftrightarrow \quad \text{MM} \lesssim \frac{1}{2\rho^2} \quad (\text{waveform error invisible})$$

Toubiana and Mark papers

$$\sigma_a = \sqrt{(\Gamma^{-1})_{aa}} \propto \frac{1}{\rho}$$

Frequency-domain waveforms and mass scaling

- Noise is naturally characterised in the frequency domain - compute likelihood, SNR in the frequency domain if noise is $\sim$ stationary.
- Fourier transform with FFT or analytically with stationary phase approximation (SPA) (slow variation of the amplitude).
- For our leading order quadrupole waveform SPA gives:

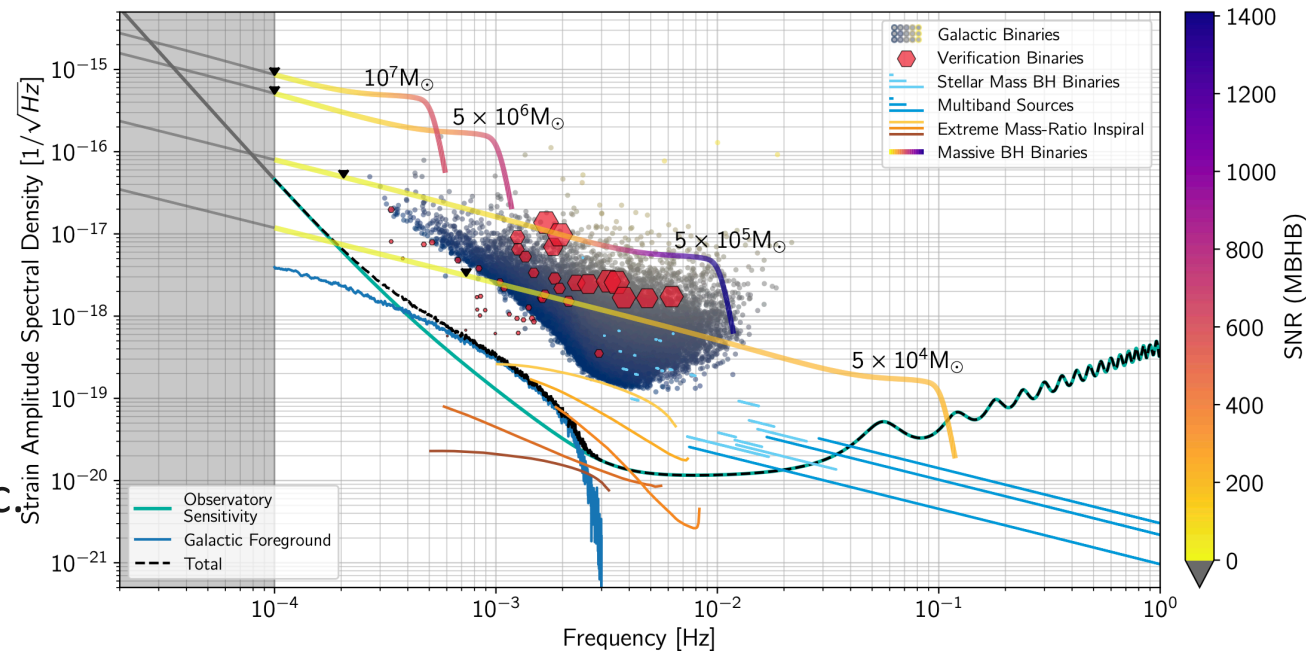
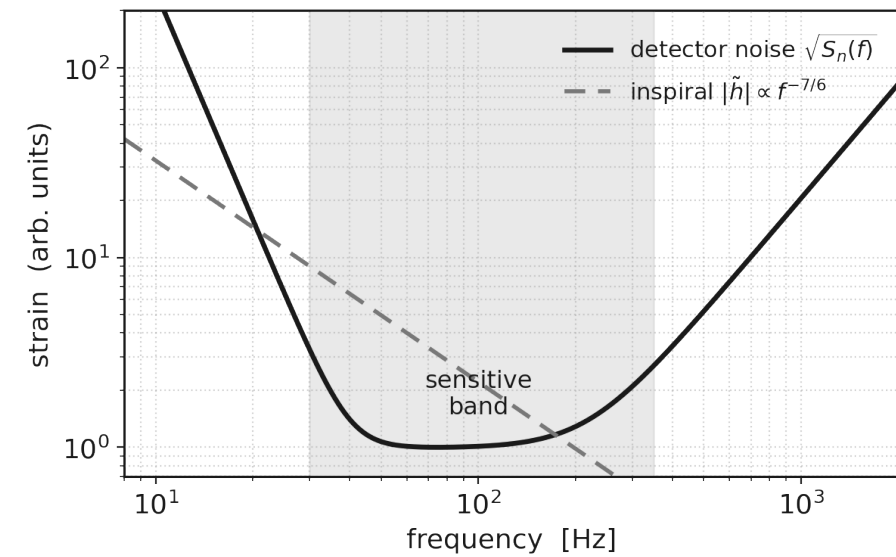
$$\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i\Psi(f)}$$

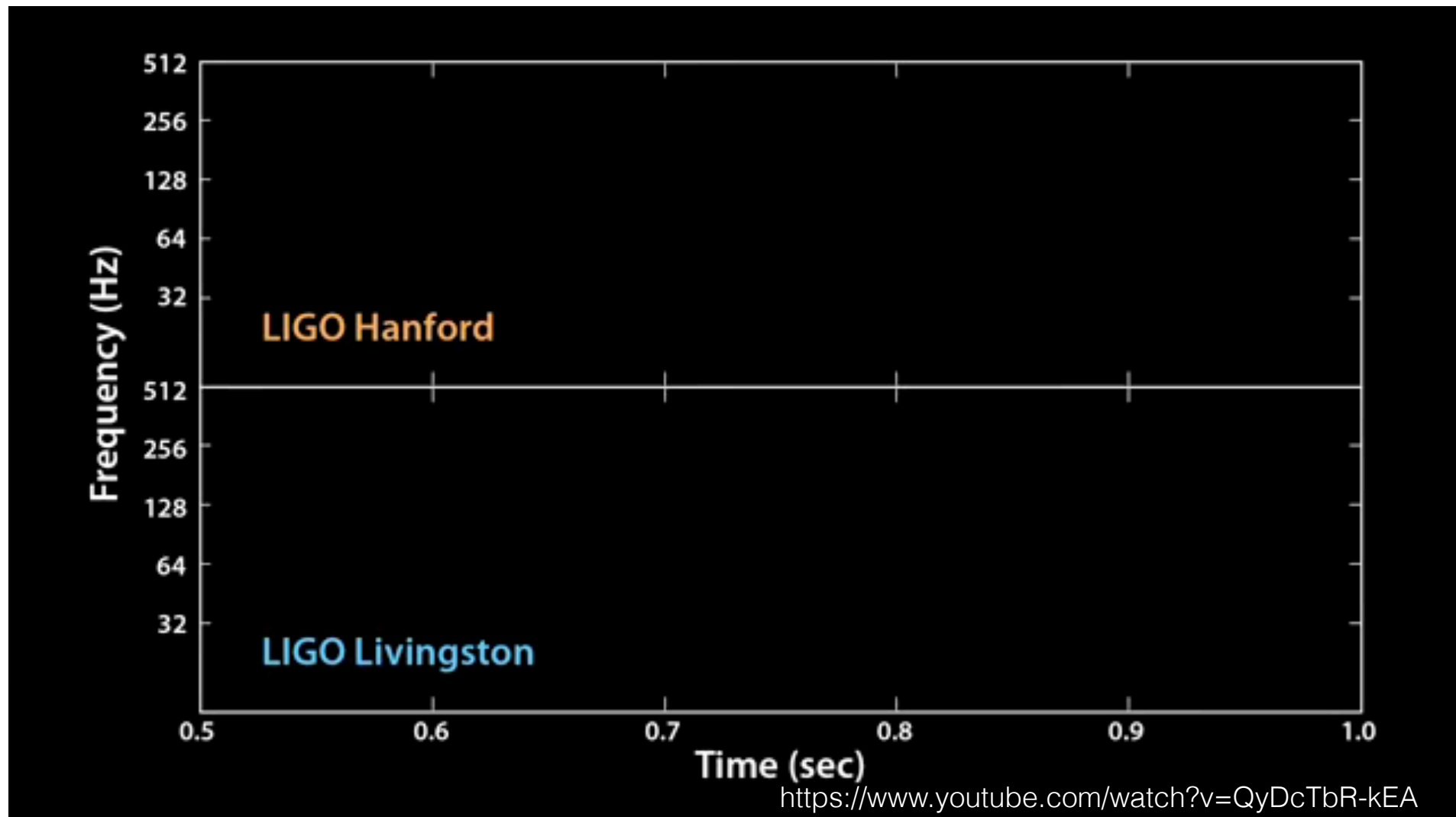
$$\Psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128} \left(\pi \frac{G\mathcal{M}}{c^3} f \right)^{-5/3}$$

- Remember - BHs: only fundamental constants = G, c .

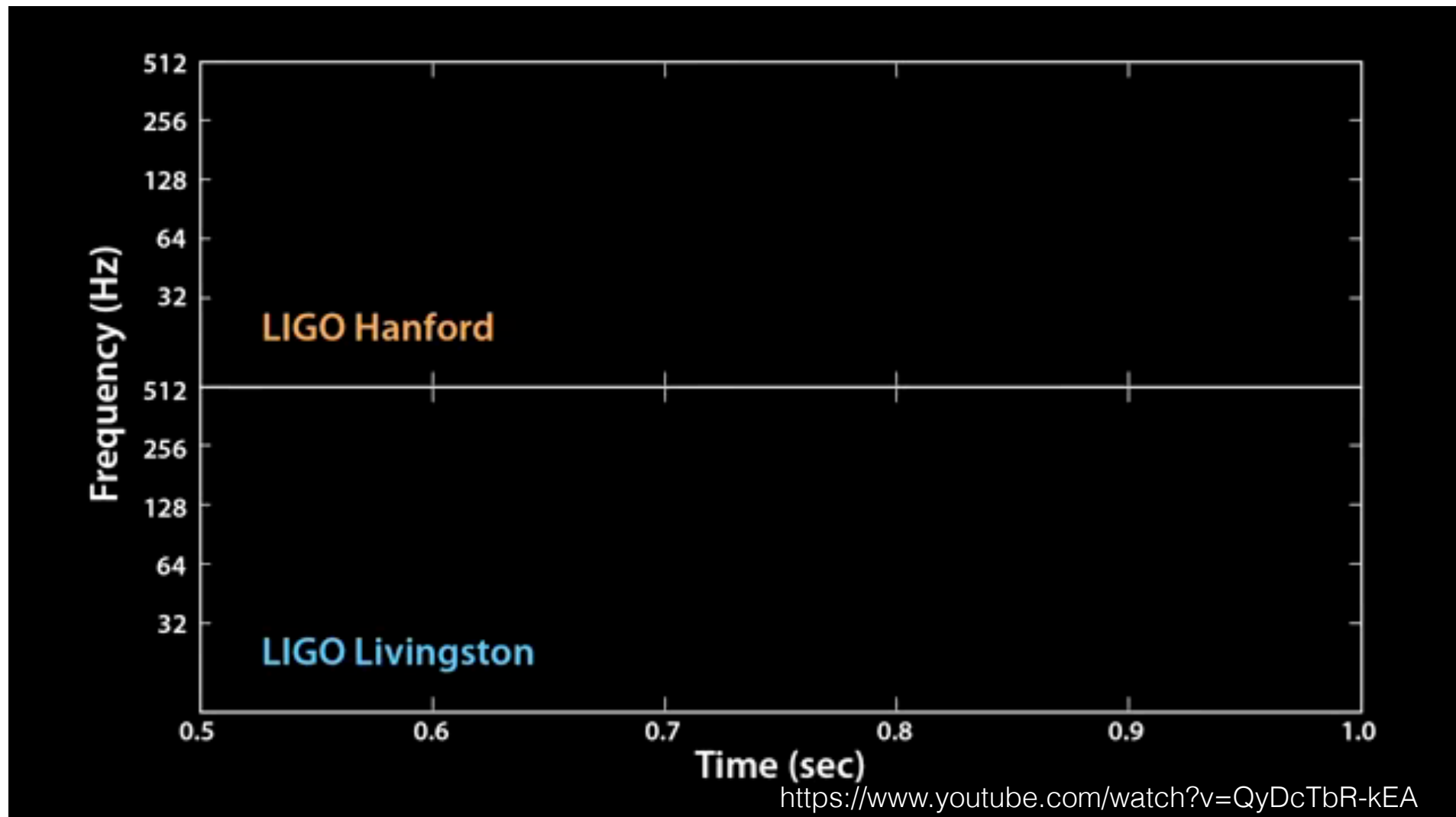
Can use mass as third fundamental unit!

Alls vacuum solutions of GR scale with mass!





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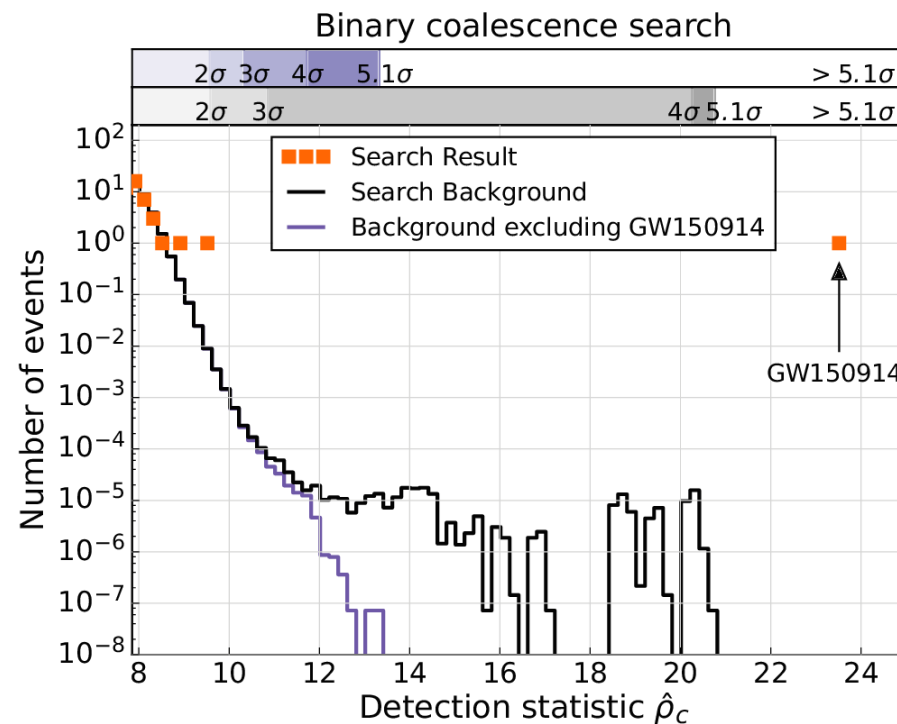
Calculation of “5.1 sigma”

- Calculate background with 0.1s time shifts (pyCBC):
16 days -> 608 000 years.
- trial factor 3 for 3 bins.
- No comparably strong events found in time shifted data:

- $\text{FAR} < 5 \times 10^{-6} \text{ yr}^{-1}$

- $\text{FAP} < 2 \times 10^{-7} \simeq 3/(16 \cdot 24 \cdot 3600 \cdot 10) \simeq 5.1\sigma$

$$5.1 = -\sqrt{2} \text{erf}^{-1}[1 - 2(1 - \text{FAP})]$$



number of time shifts

PRL116, 061102 (2016)