

Particle acceleration in astrophysics



www.cnrs.fr

Stefano Gabici
APC, Paris

gabici@apc.in2p3.fr



Plan of the lecture

The problem of particle acceleration in astrophysics (from yesterday's class)

[9] Why it is a problem to understand particle acceleration in astrophysics

Plan of the lecture

The problem of particle acceleration in astrophysics (from yesterday's class)

[9] Why it is a problem to understand particle acceleration in astrophysics

[1] How to accelerate particles in astrophysics

[2] Way out: time varying B-fields

[3] The Hillas criterion

[4] Fermi's seminal idea

The basics

Plan of the lecture

The problem of particle acceleration in astrophysics (from yesterday's class)

[9] Why it is a problem to understand particle acceleration in astrophysics

[1] How to accelerate particles in astrophysics

[2] Way out: time varying B-fields

[3] The Hillas criterion

[4] Fermi's seminal idea

The basics

Fermi II and Fermi I
(DSA) mechanisms

[5] Second order Fermi acceleration

[6] First order Fermi acceleration

[7] Shock waves in one slide

[8] Diffusive shock acceleration

Plan of the lecture

The problem of particle acceleration in astrophysics (from yesterday's class)

[9] Why it is a problem to understand particle acceleration in astrophysics

[1] How to accelerate particles in astrophysics

[2] Way out: time varying B-fields

[3] The Hillas criterion

[4] Fermi's seminal idea

The basics

Fermi II and Fermi I
(DSA) mechanisms

[5] Second order Fermi acceleration

[6] First order Fermi acceleration

[7] Shock waves in one slide

[8] Diffusive shock acceleration

Particle acceleration at relativistic shocks

[9] General considerations and simple estimates (I am not an expert!)

Way-out: time varying B

We DO need electric fields to accelerate particles!

Way-out: time varying B

We DO need electric fields to accelerate particles!

Maxwell equations

$$\nabla \vec{E} = 4\pi \rho$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Way-out: time varying B

We DO need electric fields to accelerate particles!

Maxwell equations

$$\nabla \vec{E} = 4\pi \rho = 0 \quad \rightarrow \text{plasma quasi-neutrality}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Way-out: time varying B

We DO need electric fields to accelerate particles!

Maxwell equations

$$\nabla \vec{E} = 4\pi \rho = 0 \quad \rightarrow \text{plasma quasi-neutrality}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

Faraday law

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Way-out: time varying B

We DO need electric fields to accelerate particles!

Maxwell equations

$$\nabla \vec{E} = 4\pi \rho = 0 \quad \rightarrow \text{plasma quasi-neutrality}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

Faraday law

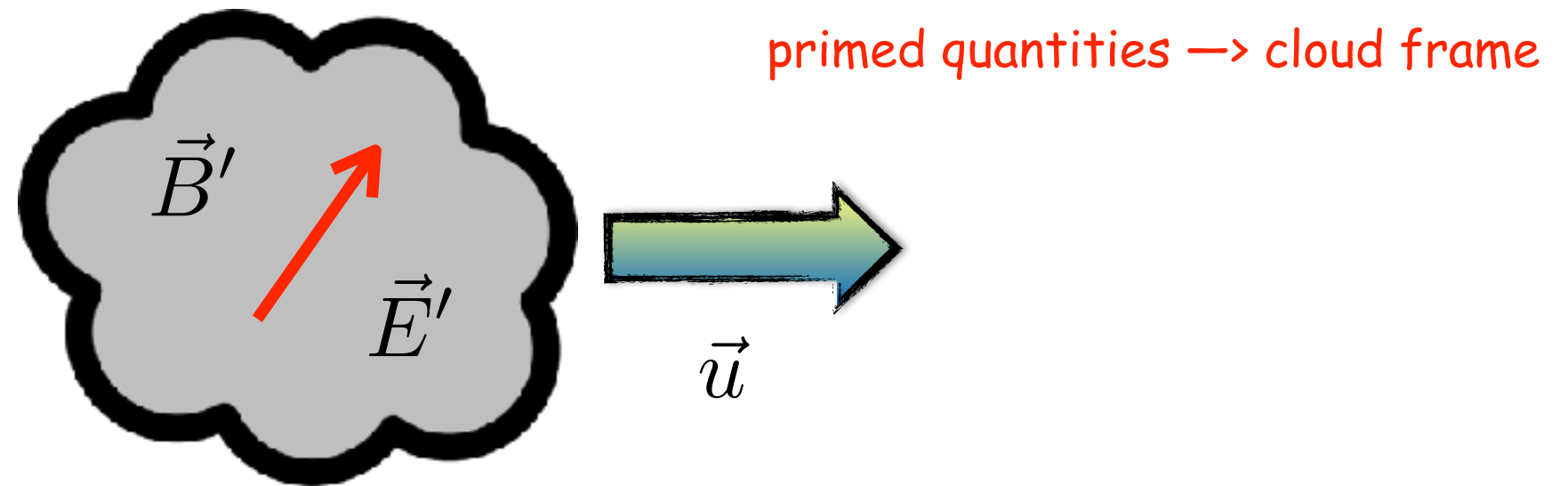
$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

A time varying magnetic field acts as a source of electric field!

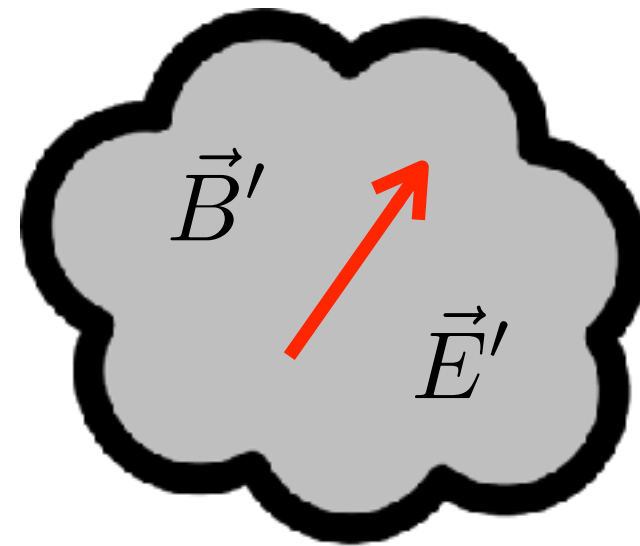
Equivalent way to see that: change rest frame

Consider a magnetised cloud of plasma moving at a (non relativistic) velocity u

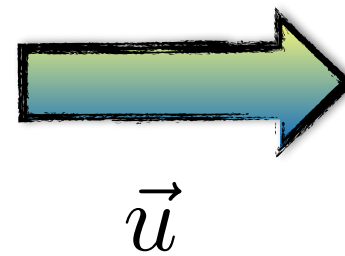


Equivalent way to see that: change rest frame

Consider a magnetised cloud of plasma moving at a (non relativistic) velocity u



primed quantities $\rightarrow$ cloud frame

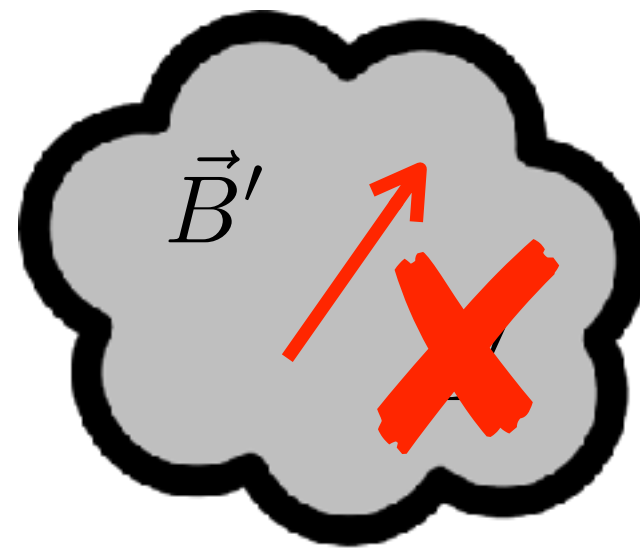


Lorentz transformation

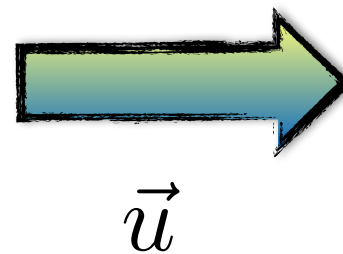
$$\vec{E}' = \vec{E} + \frac{\vec{u}}{c} \times \vec{B}$$

Equivalent way to see that: change rest frame

Consider a magnetised cloud of plasma moving at a (non relativistic) velocity u



primed quantities $\rightarrow$ cloud frame



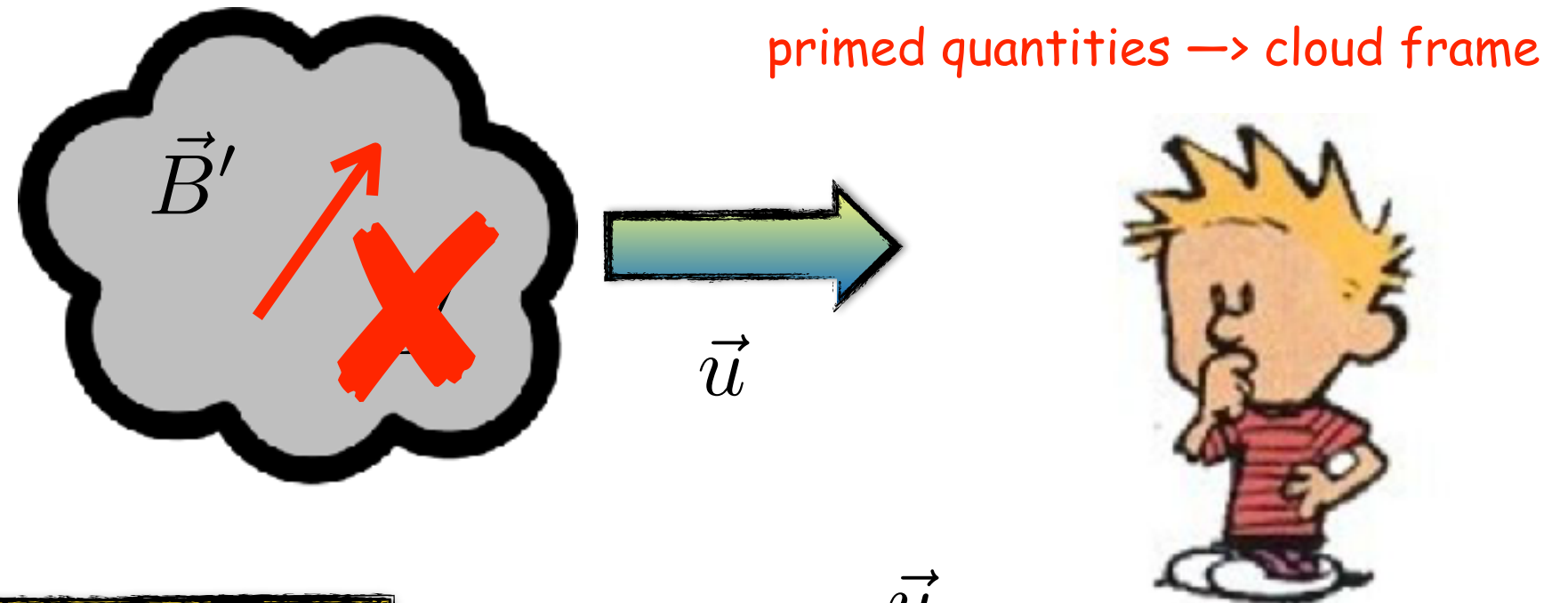
Lorentz transformation

$$\vec{E}' = \vec{E} + \frac{\vec{u}}{c} \times \vec{B}$$

$$0 = \vec{E} + \frac{\vec{u}}{c} \times \vec{B} \longrightarrow \vec{E} = -\frac{\vec{u}}{c} \times \vec{B}$$

Equivalent way to see that: change rest frame

Consider a magnetised cloud of plasma moving at a (non relativistic) velocity u



Lorentz transformation

$$\vec{E}' = \vec{E} + \frac{\vec{u}}{c} \times \vec{B}$$

$$0 = \vec{E} + \frac{\vec{u}}{c} \times \vec{B} \longrightarrow \boxed{\vec{E} = -\frac{\vec{u}}{c} \times \vec{B}}$$

an observer in the lab frame sees an electric field!

Order of magnitude estimates of the induced electric field

time-varying B-field

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

Order of magnitude estimates of the induced electric field

time-varying B-field

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

characteristic length

$$\nabla \times \rightarrow \frac{1}{L}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{T}$$

characteristic time

Order of magnitude estimates of the induced electric field

time-varying B-field

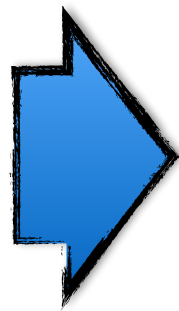
$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

characteristic length

$$\nabla \times \rightarrow \frac{1}{L}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{T}$$

characteristic time



$$E \approx \frac{L}{T} \frac{B}{c} \approx \frac{U}{c} B$$

characteristic velocity

Order of magnitude estimates of the induced electric field

time-varying B-field

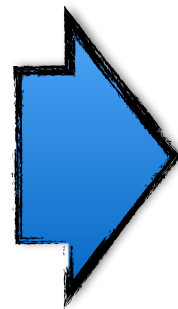
$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

characteristic length

$$\nabla \times \rightarrow \frac{1}{L}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{T}$$

characteristic time

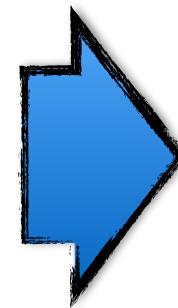


$$E \approx \frac{L}{T} \frac{B}{c} \approx \frac{U}{c} B$$

characteristic velocity

Lorentz
transformation

$$\vec{E} = -\frac{\vec{u}}{c} \times \vec{B}$$



$$E \approx \frac{U}{c} B$$



The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

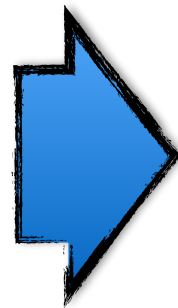


The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

$$E \approx \frac{U}{c} B$$



$$\mathcal{E}_{max} = \frac{q}{c} U B L$$

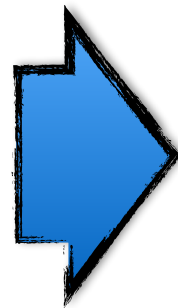


The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

$$E \approx \frac{U}{c} B$$



$$\mathcal{E}_{max} = \frac{q}{c} U B L$$

electric charge

velocity

B-field

size

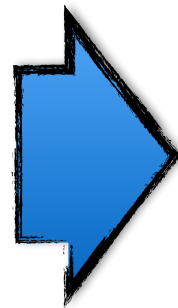


The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

$$E \approx \frac{U}{c} B$$



$$\mathcal{E}_{max} = \frac{q}{c} U B L$$

electric charge $\rightarrow q$

velocity $\rightarrow U$

B-field $\rightarrow B$

size $\rightarrow L$

acceleration rate $\rightarrow$

$$\frac{d\mathcal{E}}{dt} = c \frac{d\mathcal{E}}{dx} = c q E = q U B$$

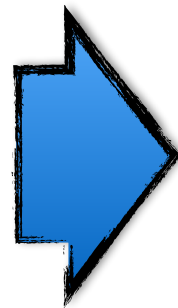


The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

$$E \approx \frac{U}{c} B$$



$$\mathcal{E}_{max} = \frac{q}{c} U B L$$

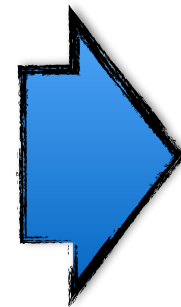
electric charge $\rightarrow q$
velocity $\rightarrow U$
B-field $\rightarrow B$
size $\rightarrow L$

acceleration rate $\rightarrow$

$$\frac{d\mathcal{E}}{dt} = c \frac{d\mathcal{E}}{dx} = c q E = q U B$$

Larmor radius $\rightarrow$

$$R_L = \frac{\mathcal{E}}{qB}$$



$$R_L^{max} = \left(\frac{U}{c} \right) L$$

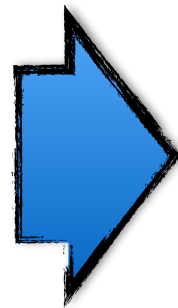


The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

$$E \approx \frac{U}{c} B$$



$$\mathcal{E}_{max} = \frac{q}{c} U B L$$

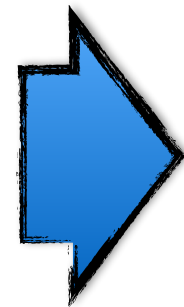
electric charge $\rightarrow q$
velocity $\rightarrow U$
B-field $\rightarrow B$
size $\rightarrow L$

acceleration rate $\rightarrow$

$$\frac{d\mathcal{E}}{dt} = c \frac{d\mathcal{E}}{dx} = c q E = q U B$$

Larmor radius $\rightarrow$

$$R_L = \frac{\mathcal{E}}{qB}$$



$$R_L^{max} = \left(\frac{U}{c} \right) L$$

acceleration time $\rightarrow$

$$\tau_{acc}^H = \frac{\mathcal{E}_{max}}{d\mathcal{E}/dt} = \left(\frac{U}{c} \right)^{-1} \frac{R_L}{c}$$



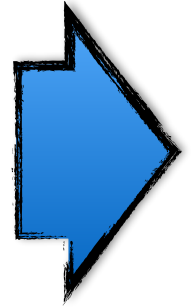
The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q E L$$

$$E \approx \frac{U}{c} B$$

electric charge
velocity
B-field
size



$$\mathcal{E}_{max} = q U B L$$

acceleration rate

$$\frac{d\mathcal{E}}{dt} = c \frac{d\mathcal{E}}{dx} = c q E = q U B$$

very general, we didn't assume anything about the nature of the accelerator!

radius →

$$R_L = \frac{\mathcal{E}}{q B}$$



$$R_L^{max} = \left(\frac{U}{c} \right) L$$

acceleration time →

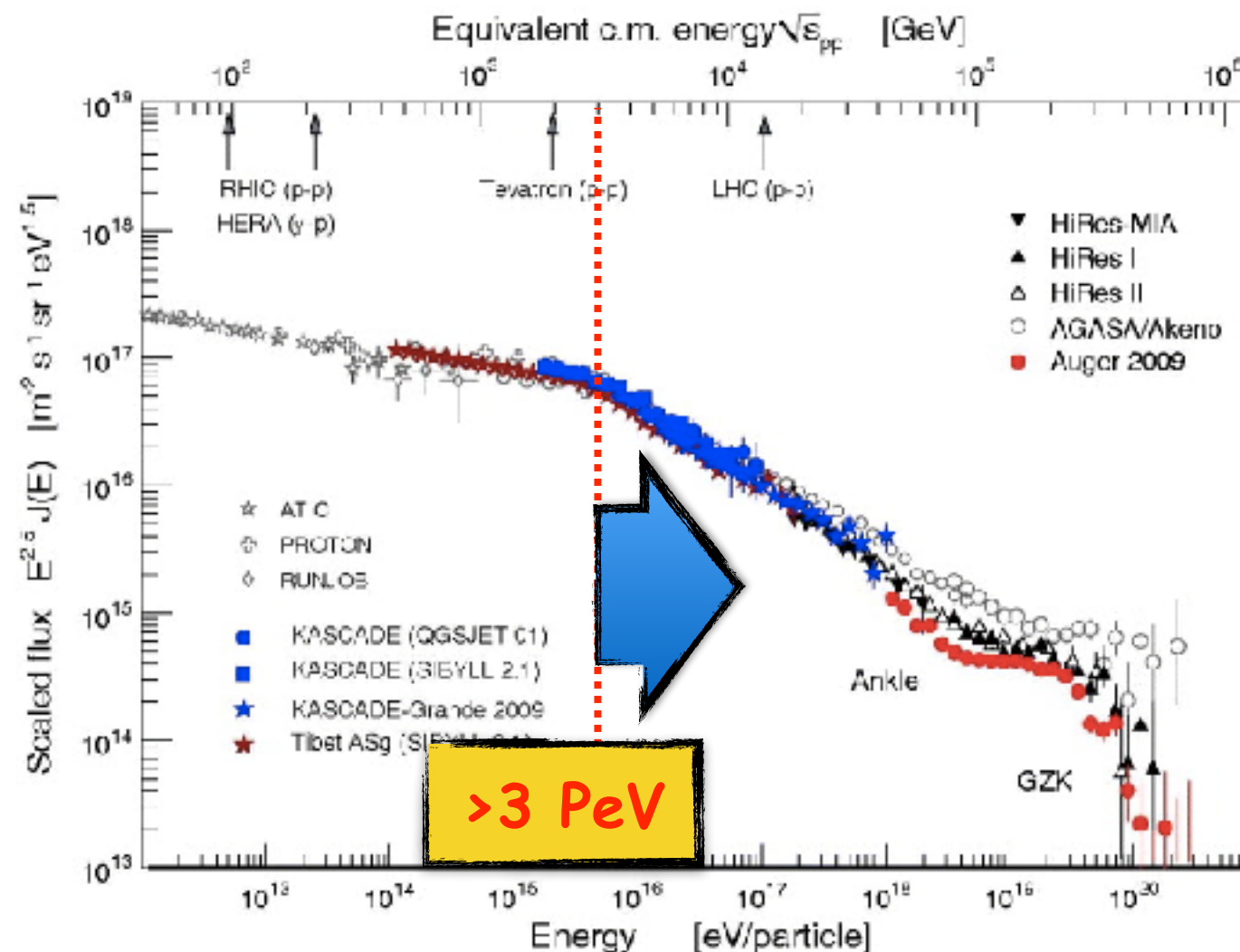
$$\tau_{acc}^H = \frac{\mathcal{E}_{max}}{d\mathcal{E}/dt} = \left(\frac{U}{c} \right)^{-1} \frac{R_L}{c}$$

The Hillas criterion in numbers

$$\mathcal{E}_{max} = \frac{q}{c} U B L \sim 3 Z \left(\frac{U}{1000 \text{ km/s}} \right) \left(\frac{B}{\mu\text{G}} \right) \left(\frac{L}{\text{pc}} \right) \text{TeV}$$

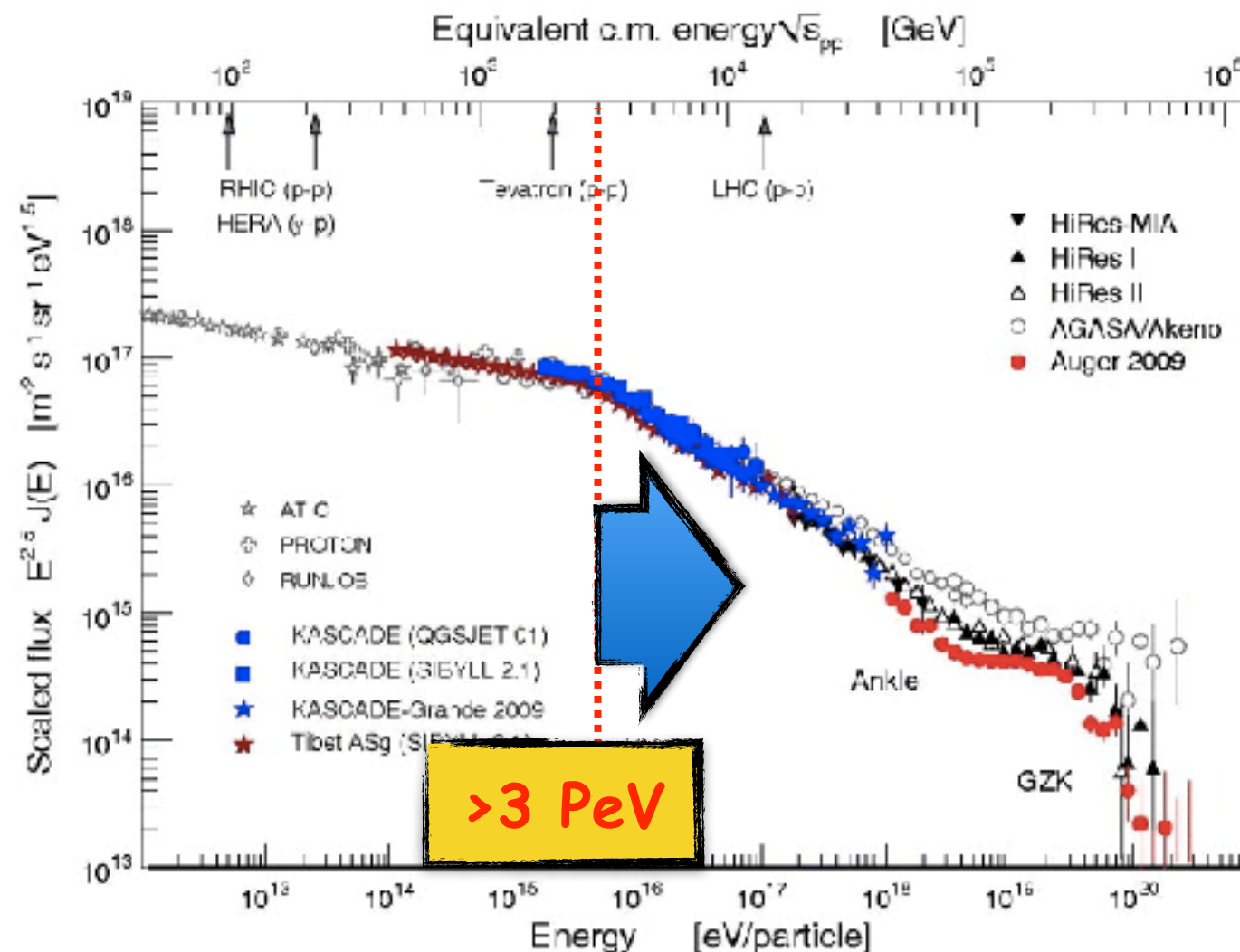
The Hillas criterion in numbers

$$\mathcal{E}_{max} = \frac{q}{c} U B L \sim 3 Z \left(\frac{U}{1000 \text{ km/s}} \right) \left(\frac{B}{\mu\text{G}} \right) \left(\frac{L}{\text{pc}} \right) \text{TeV}$$



The Hillas criterion in numbers

$$\mathcal{E}_{max} = \frac{q}{c} U B L \sim 3 Z \left(\frac{U}{1000 \text{ km/s}} \right) \left(\frac{B}{\mu\text{G}} \right) \left(\frac{L}{\text{pc}} \right) \text{TeV}$$



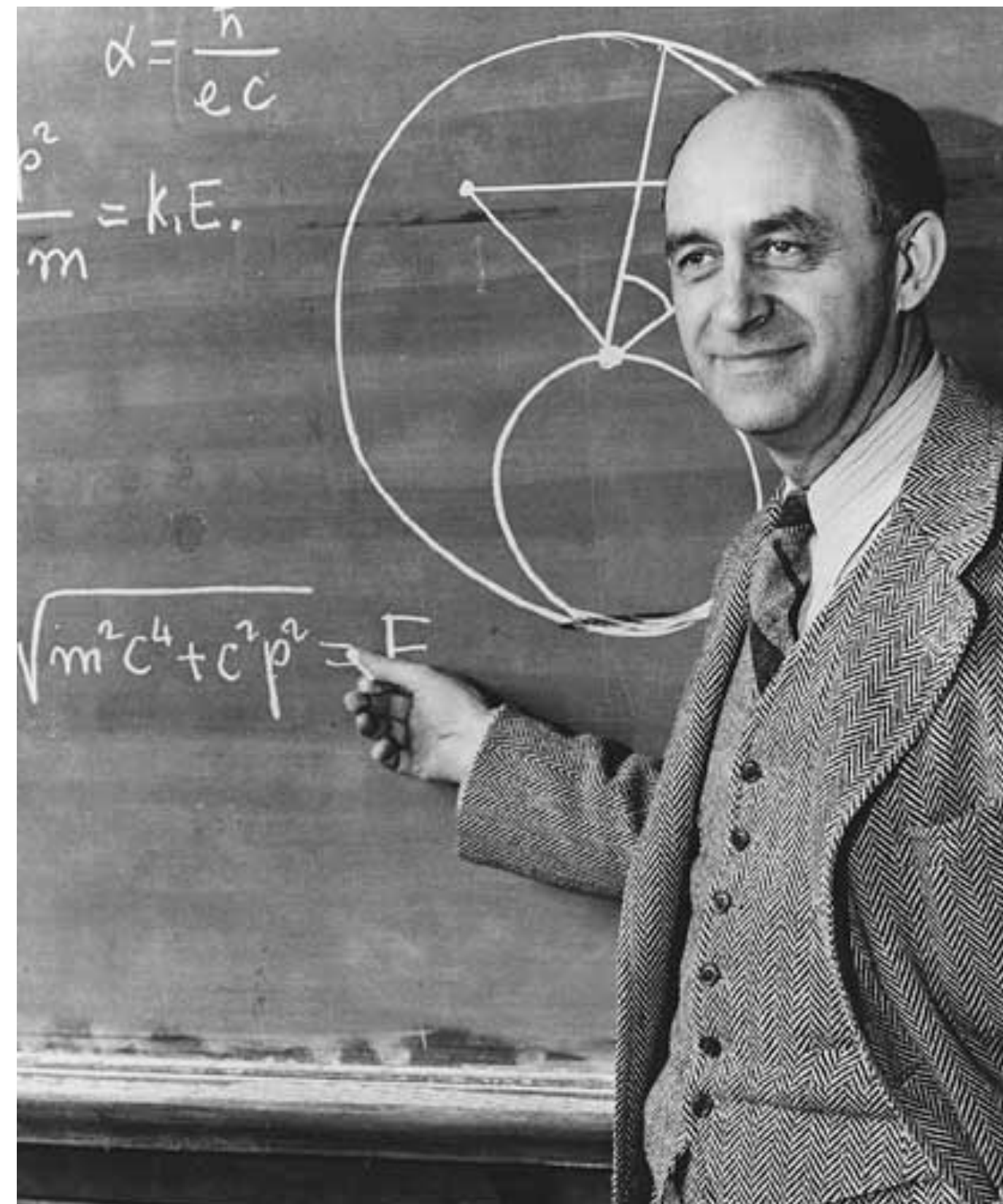
$$\tau_{acc} \sim \frac{\mathcal{E}}{d\mathcal{E}/dt} \sim \left(\frac{U}{c} \right)^{-1} \frac{R_L}{c} = 3 \left(\frac{L}{\text{pc}} \right) \text{yr}$$

Advise...

If you had to remember only one thing about cosmic ray acceleration, that would probably be the Hillas criterion. However, while this criterion imposes necessary conditions to accelerate particles, it tells us **NOTHING** about **HOW** particles are accelerated...

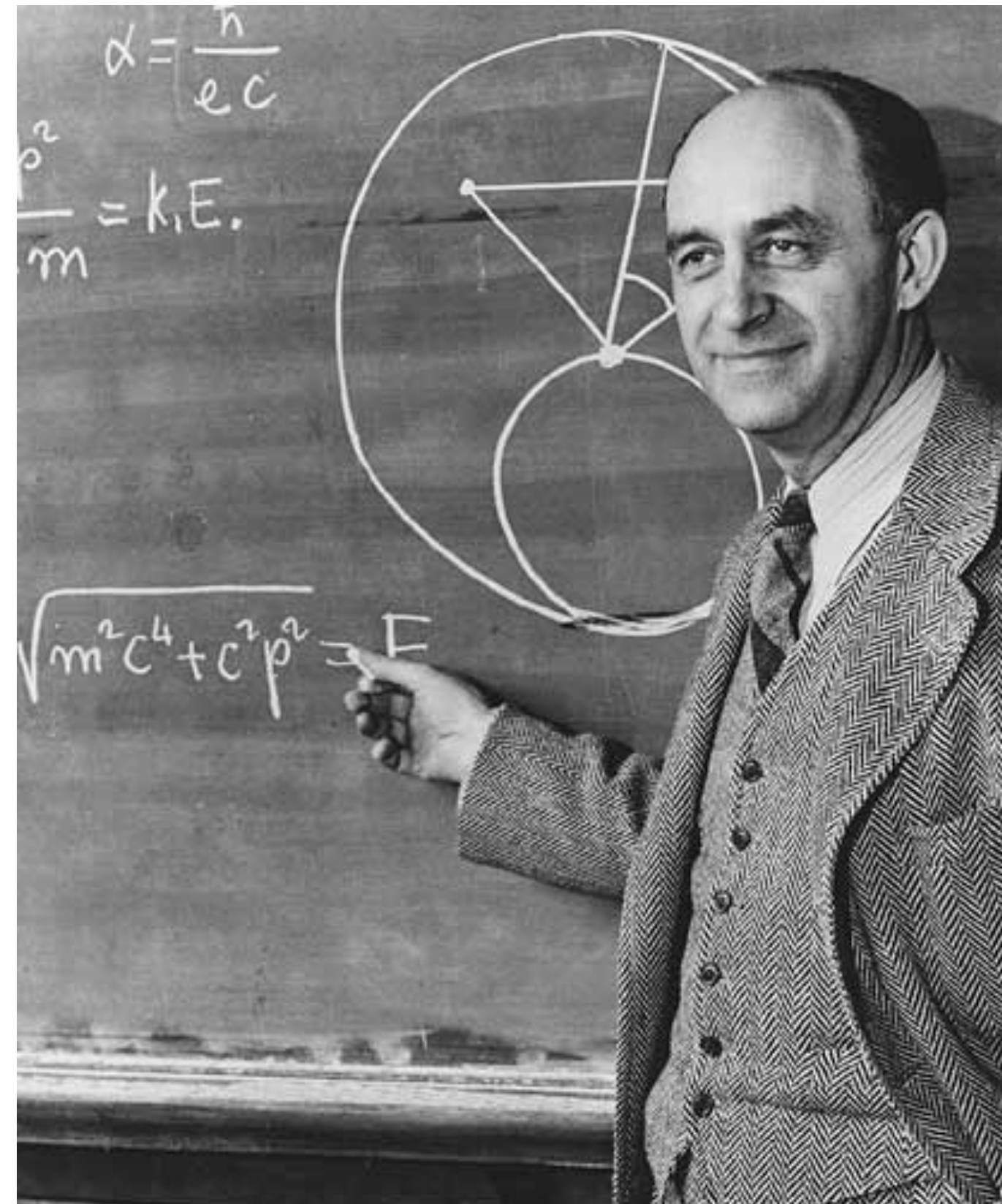
Note: from now on we indicate particle energies with E rather $\mathcal{E}$ as there is no longer ambiguity with electric field

Fermi idea



Fermi idea

■ The interstellar medium is:

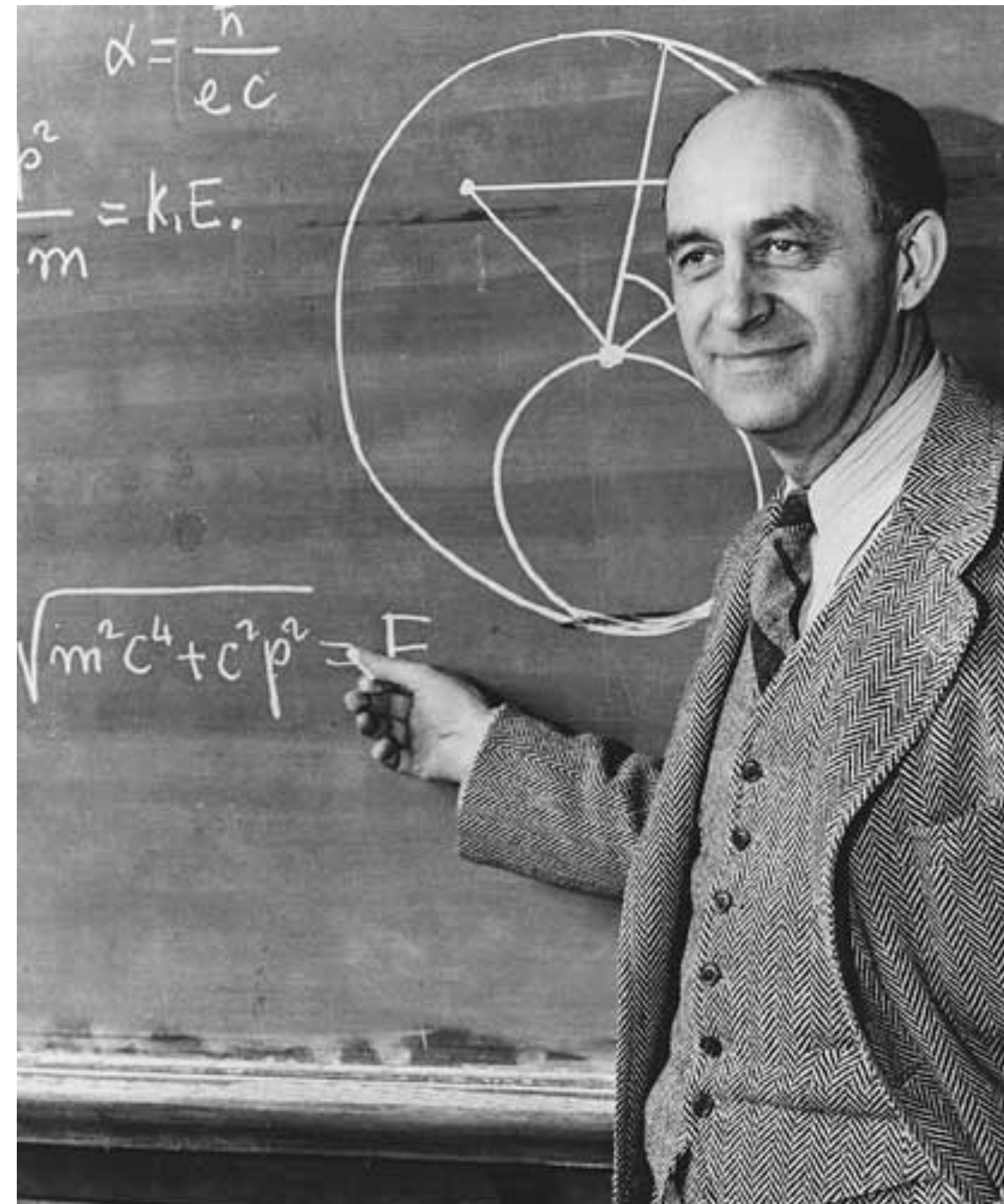


Fermi idea

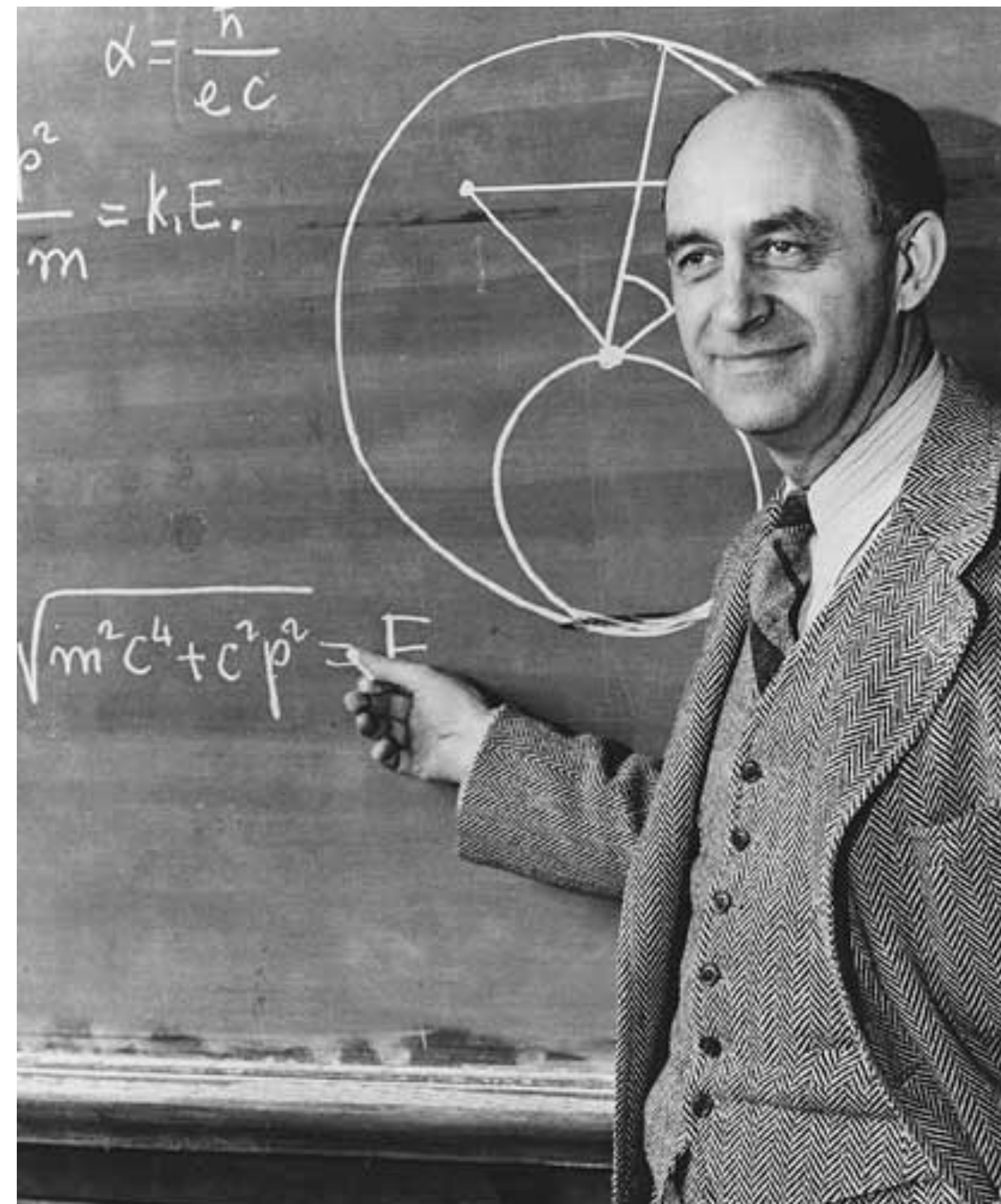
- The interstellar medium is:
 - ▶ inhomogeneous



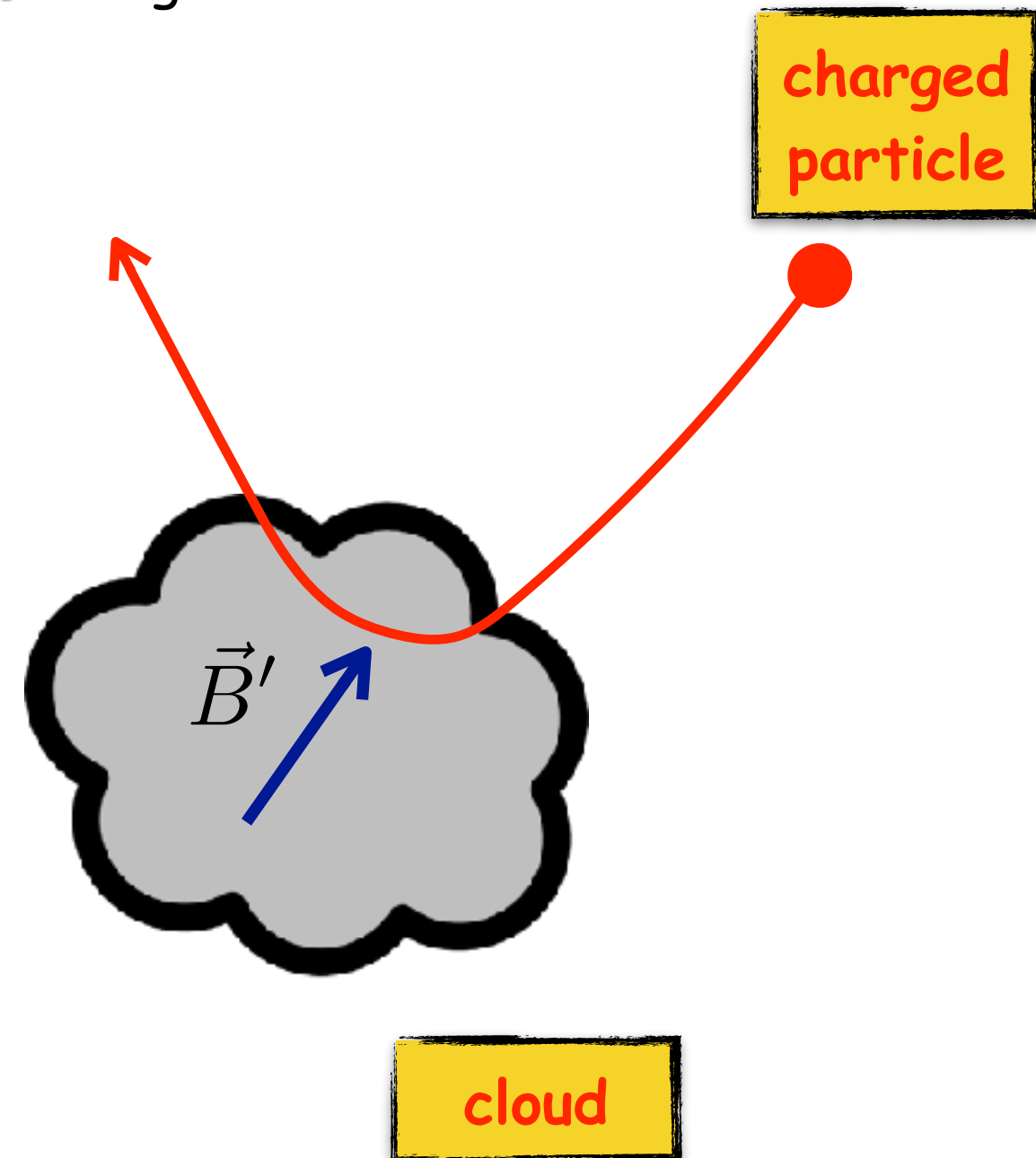
cloud



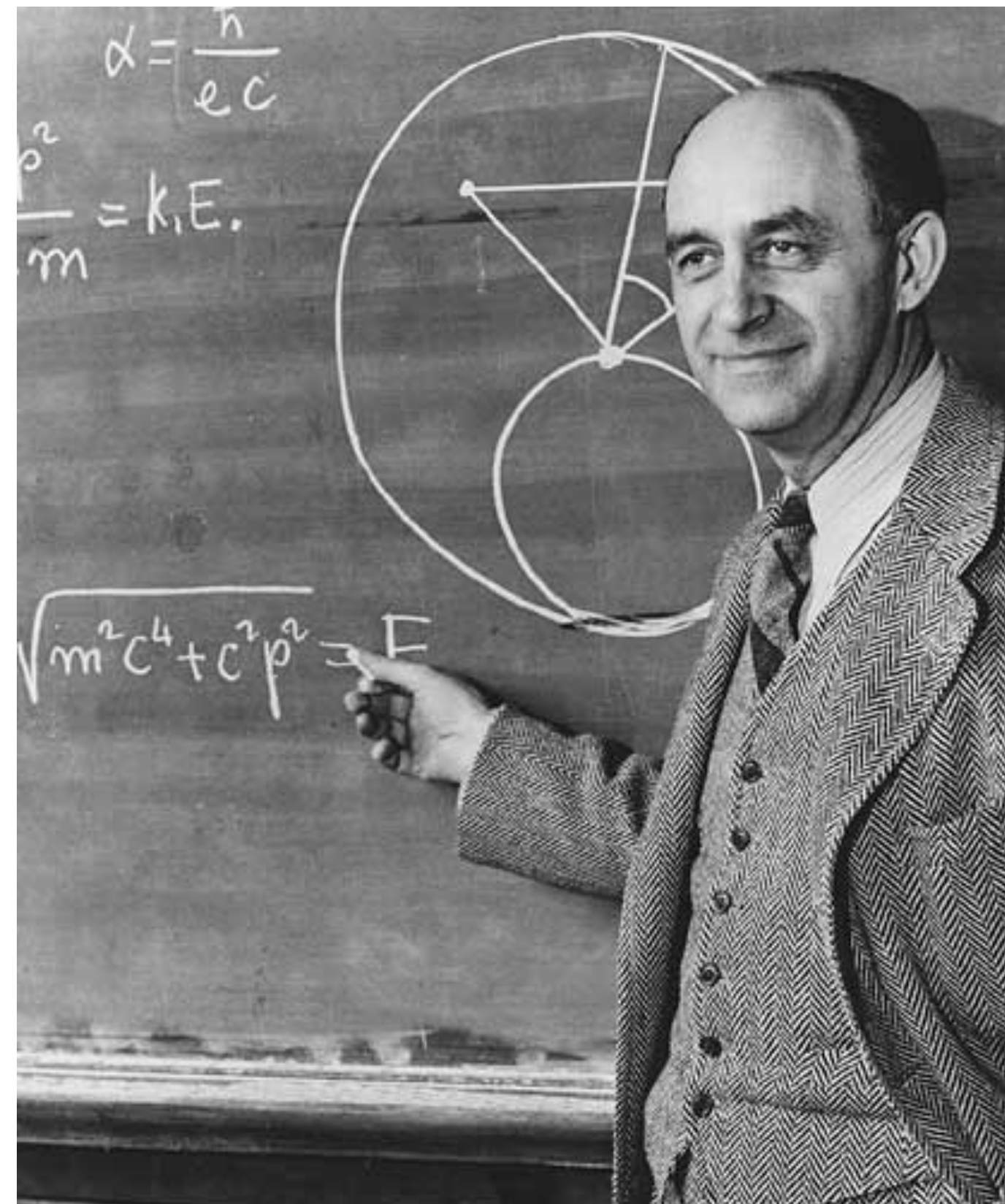
Fermi idea



- The interstellar medium is:
 - ▶ inhomogeneous
 - ▶ magnetised

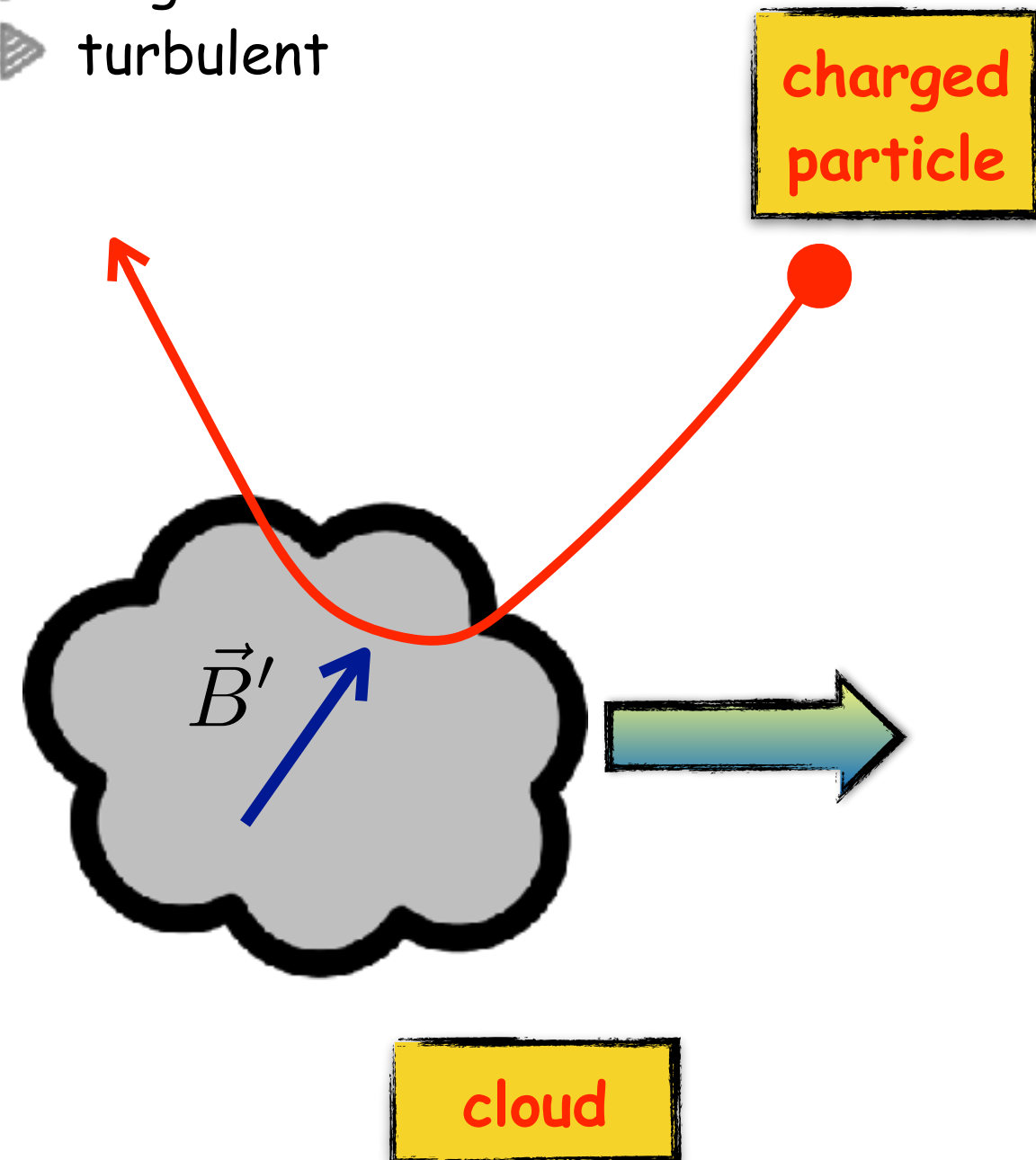


Fermi idea



■ The interstellar medium is:

- ▶ inhomogeneous
- ▶ magnetised
- ▶ turbulent



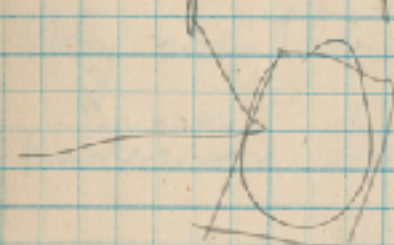
Fermi's notebook
December 1948

Dec. 4 1948

137

Theory of cosmic rays

a) Energy acquired in collisions against cosmic magnetic fields



Non relativistic case

$$M V^2$$

(M = mass of particle V = velocity of moving field)

(Proof: Head on collision gives energy gain

$$\frac{M}{2} (v + 2V)^2 - \frac{M v^2}{2} = \frac{M}{2} (4vV + 4V^2) =$$

$$= M(2vV + 2V^2) \quad \text{Prob.} = \frac{v+V}{2v}$$

Running after collision (prob. = $\frac{v-V}{2v}$) gives energy gain
 $M(-2vV + 2V^2)$

Average gain order

$$M V^2$$

Relativistic: order

$$\omega p^2$$

From Fermi's notebook

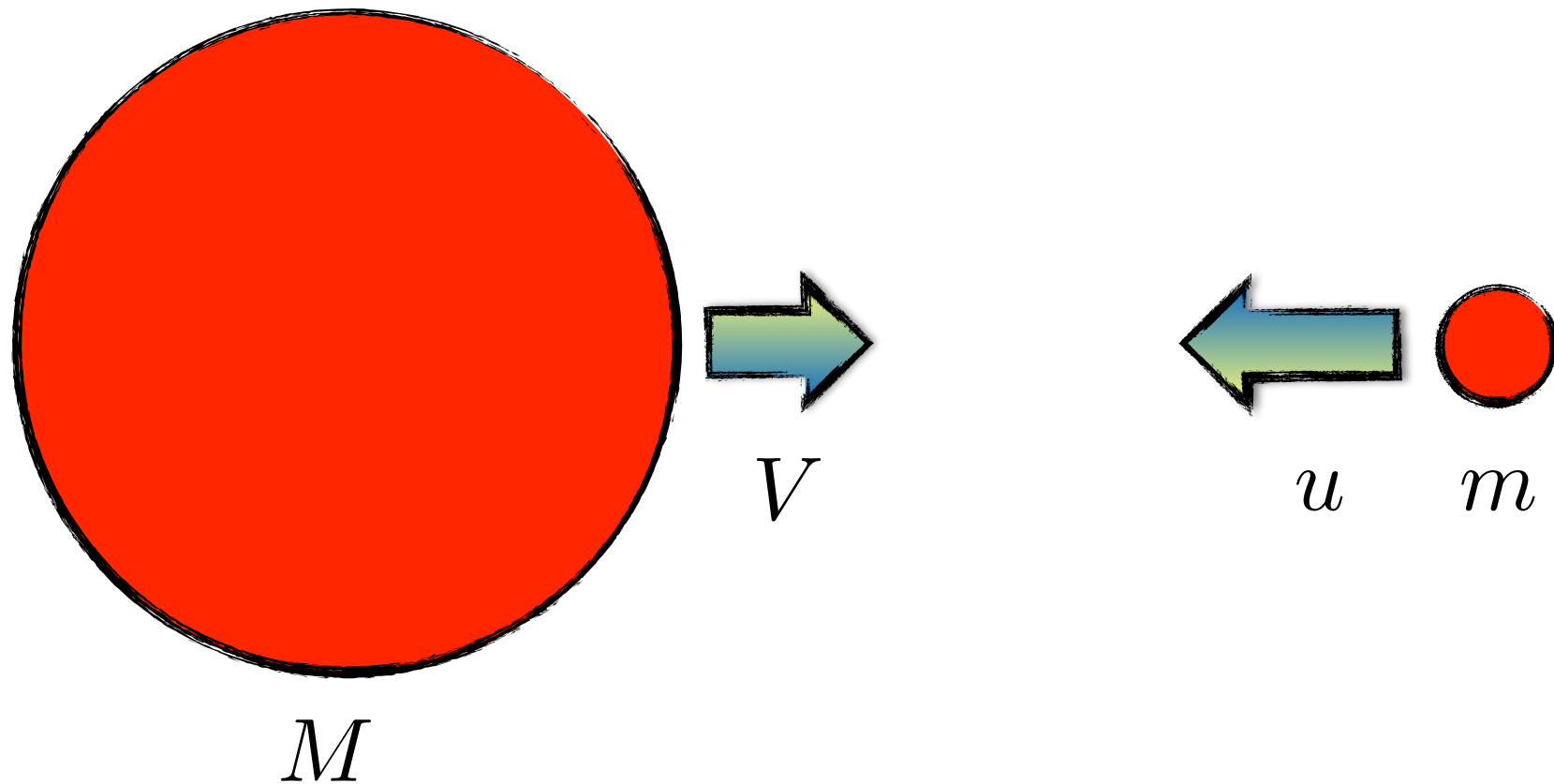
Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision

$$M \gg m$$

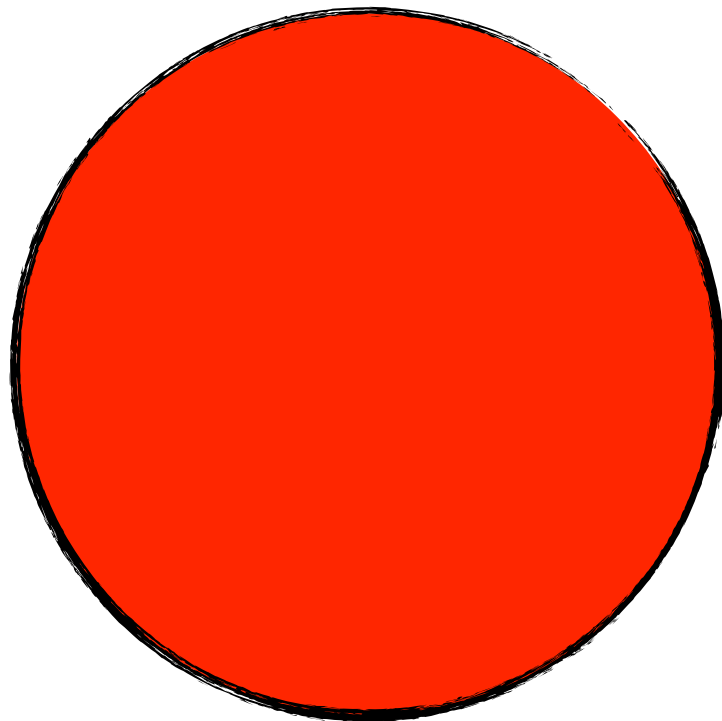


From Fermi's notebook

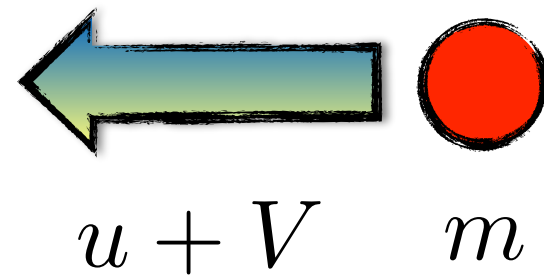
Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision

$$M \gg m$$

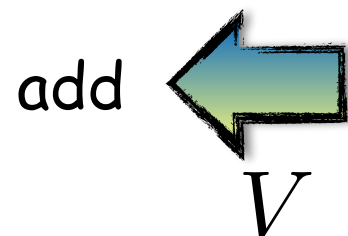


M



$u + V$

m



add

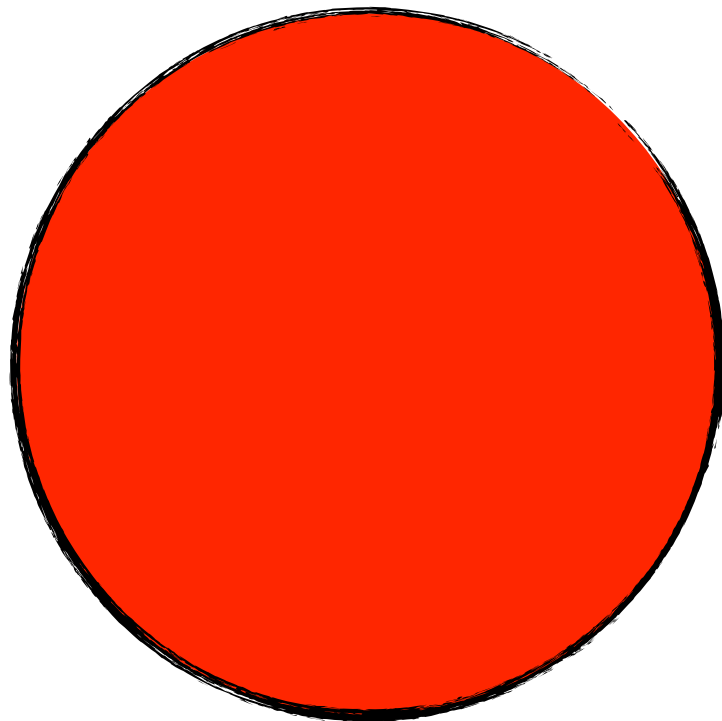
to move to the rest frame where M is at rest

From Fermi's notebook

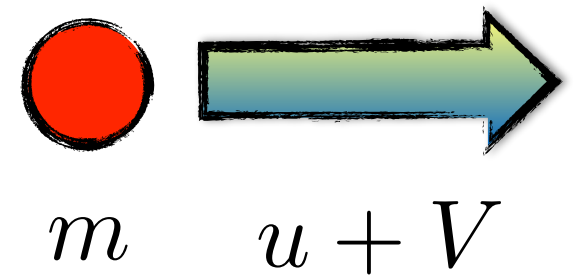
Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision

$$M \gg m$$



M



$m \quad u + V$

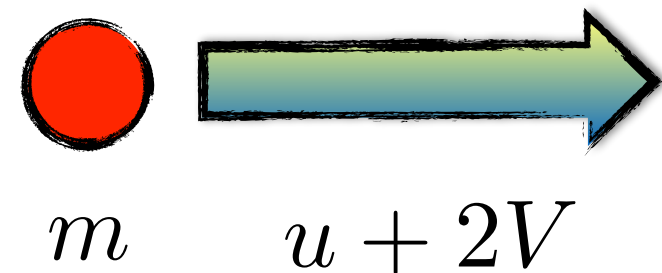
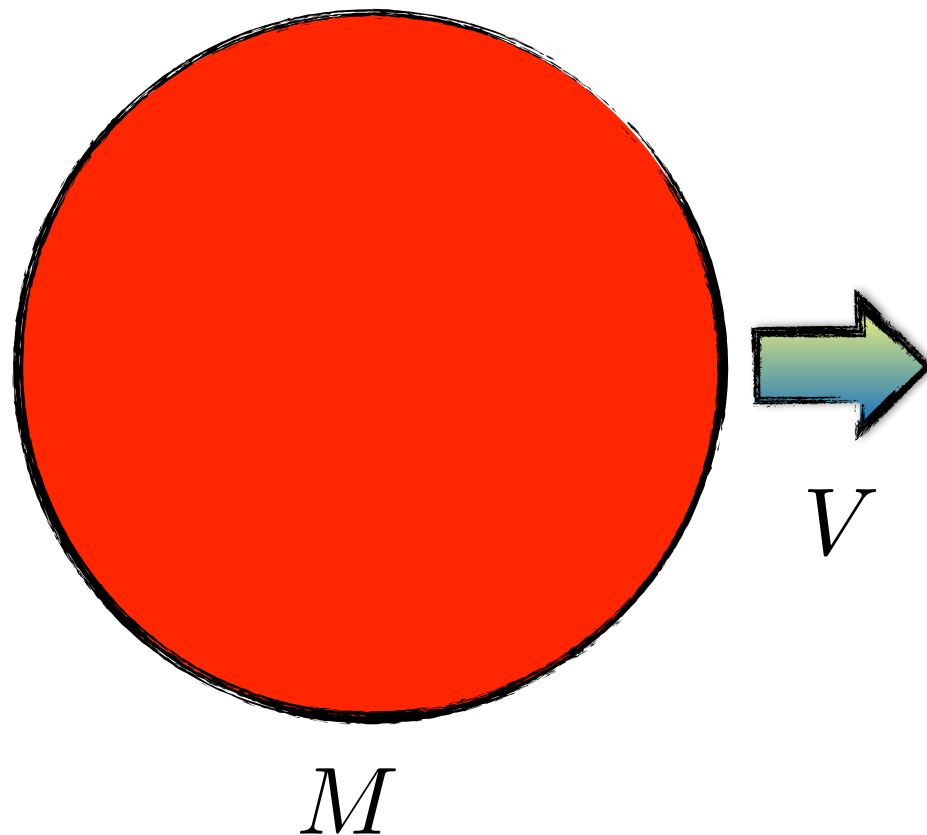
elastic collision: same velocity but in the opposite direction

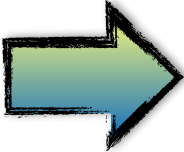
From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision

$$M \gg m$$



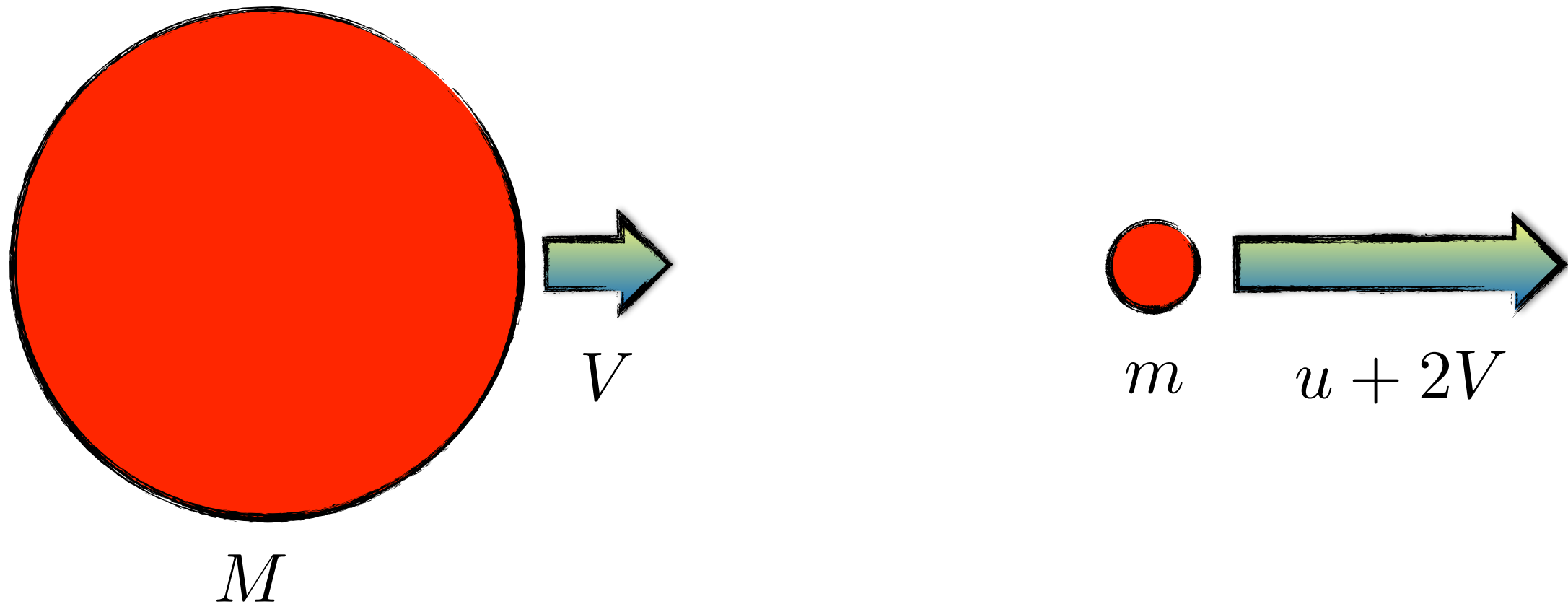
add  to move back to the lab rest frame
 V

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision

$$M \gg m$$



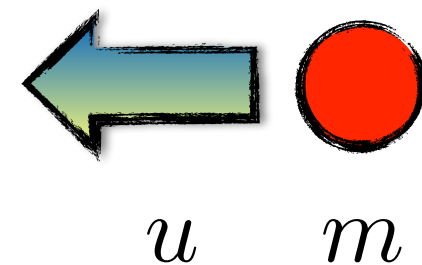
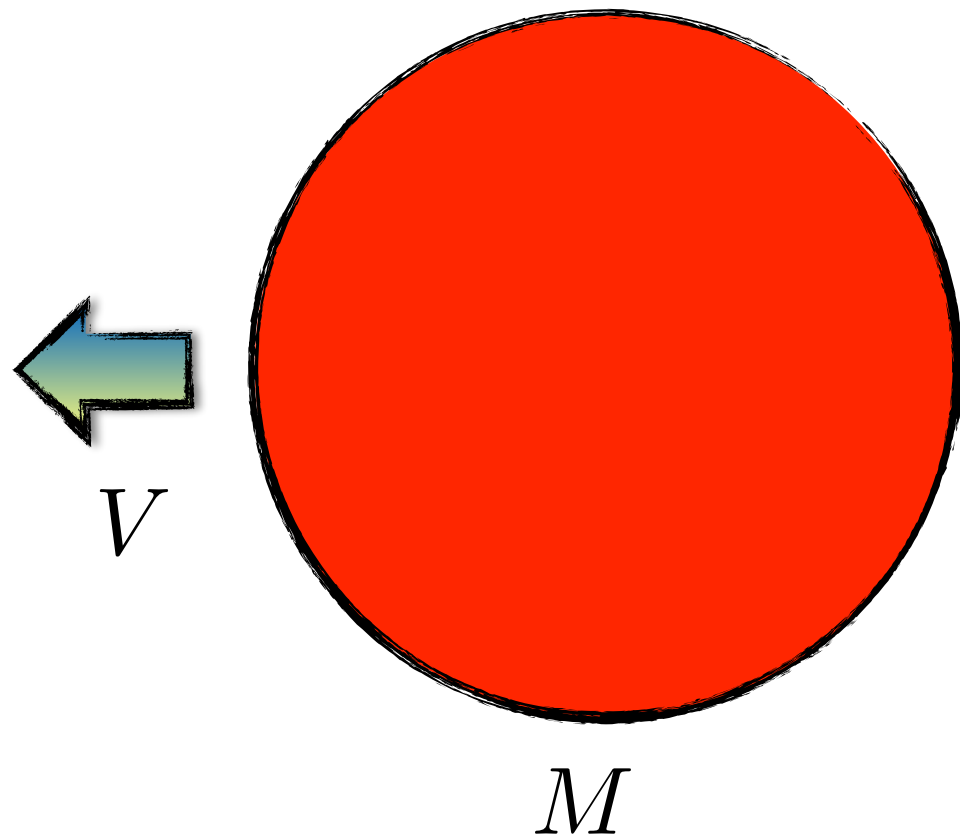
Energy gain $\rightarrow \frac{1}{2}m(u + 2V)^2 - \frac{1}{2}mu^2 = 2m(V^2 + uV)$

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

running after collision

$$M \gg m$$

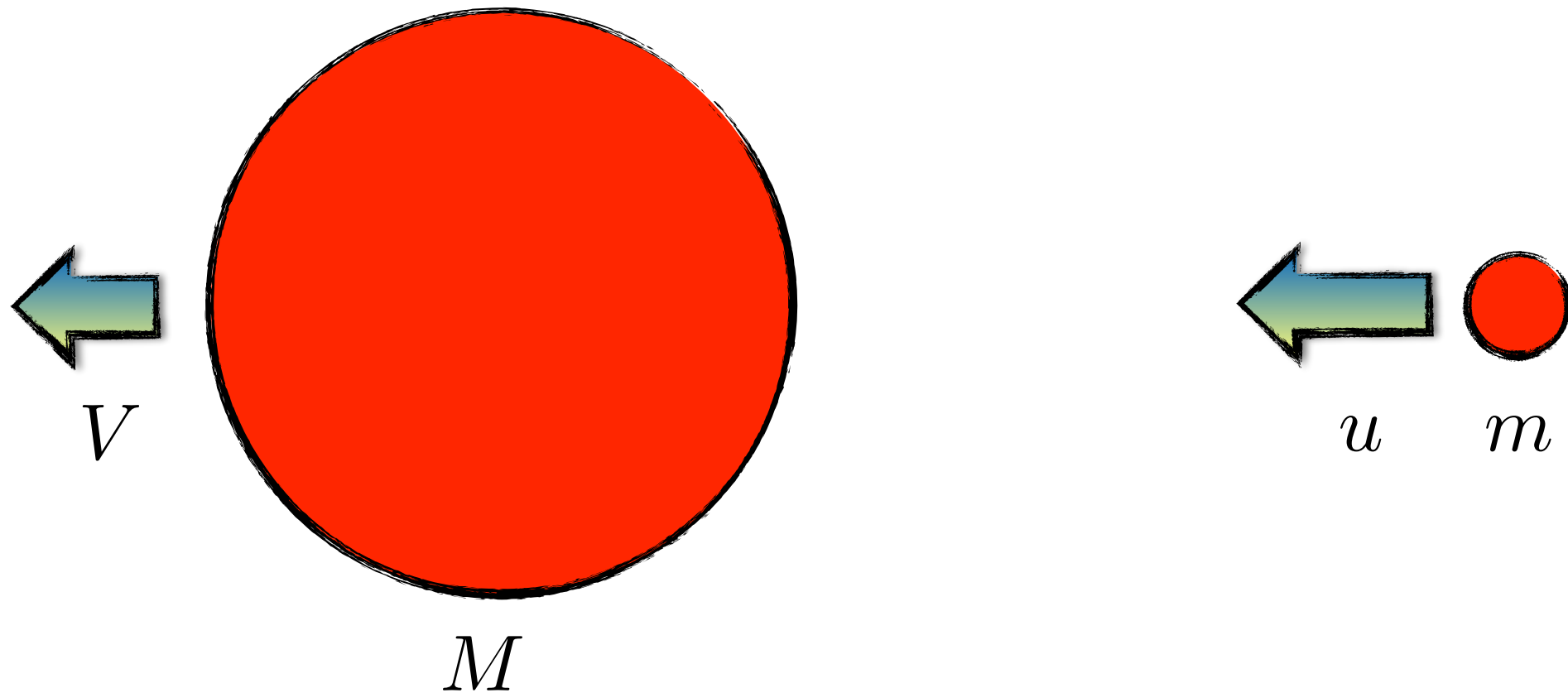


From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

running after collision

$$M \gg m$$



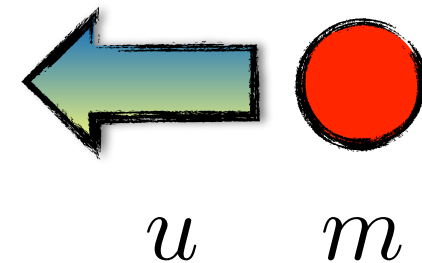
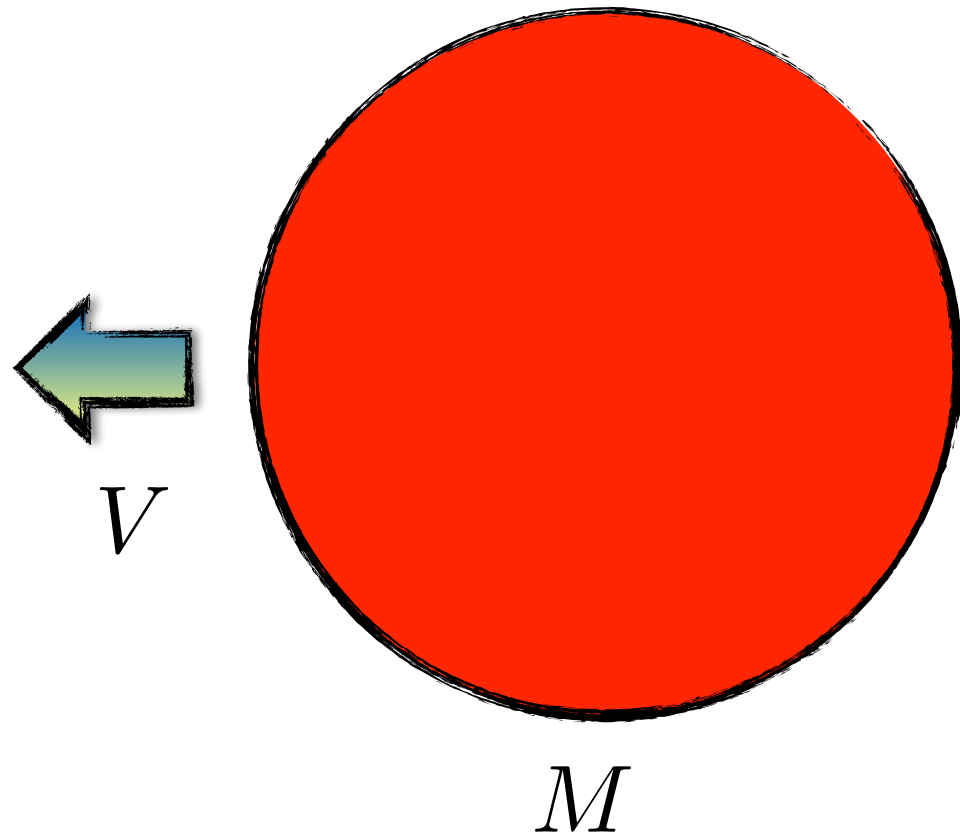
Energy gain $\rightarrow \frac{1}{2}m(u - 2V)^2 - \frac{1}{2}mu^2 = 2m(V^2 - uV)$

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

running after collision

$$M \gg m$$

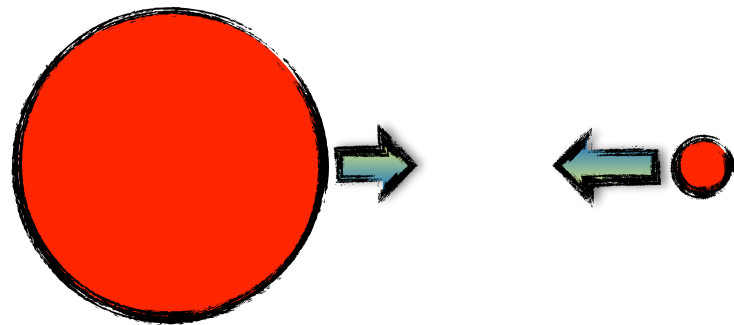


Energy loss $\frac{1}{2}m(u - 2V)^2 - \frac{1}{2}mu^2 = 2m(V^2 - uV) < 0$

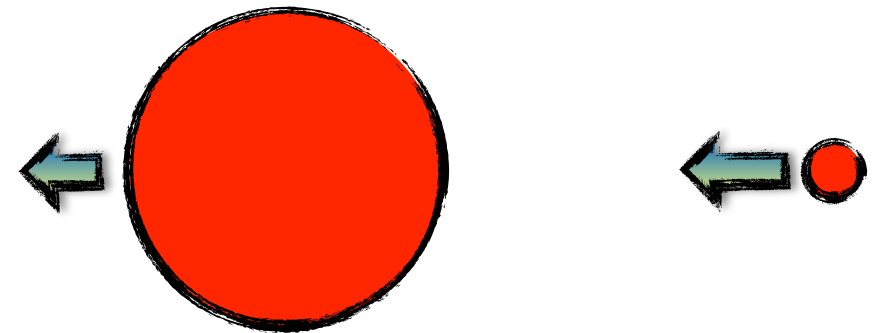
From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision -> gain



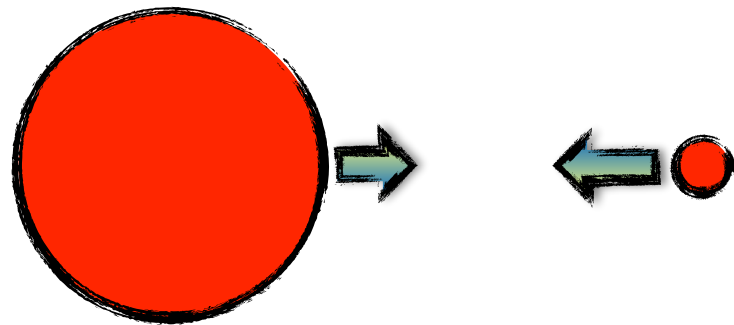
running after collision -> loss



From Fermi's notebook

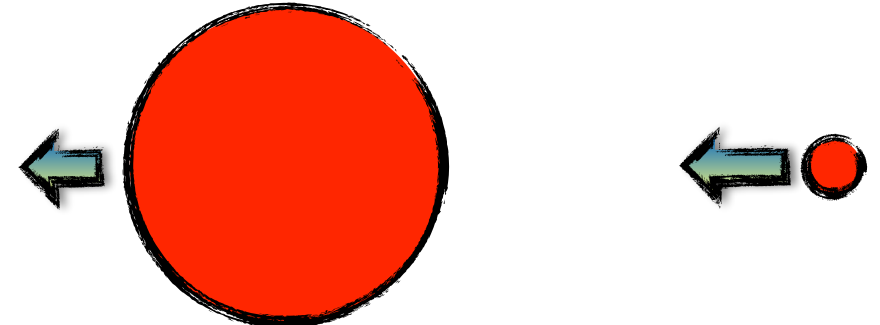
Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision -> gain



$$\frac{u + V}{2u}$$

running after collision -> loss



$$\frac{u - V}{2u}$$

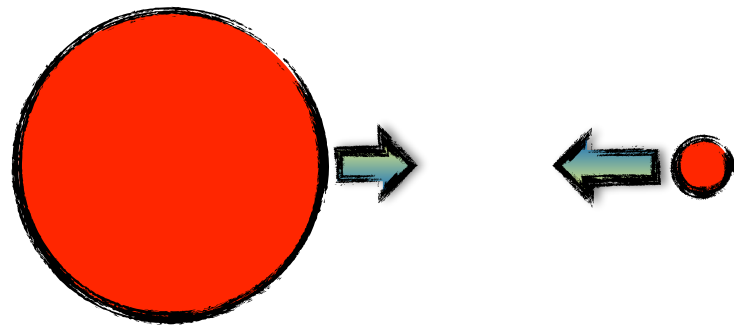
probability

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

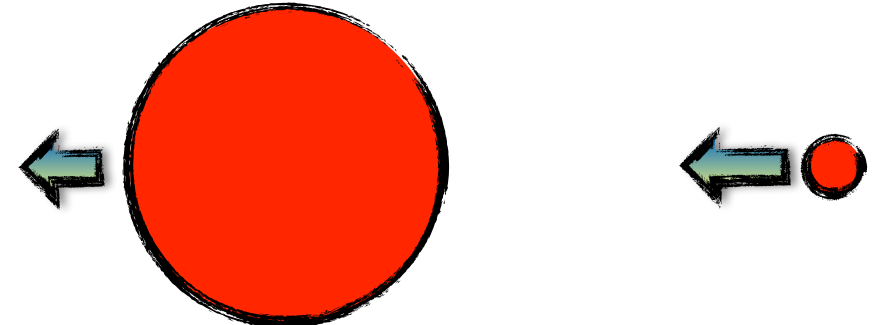
head-on collision -> gain

running after collision -> loss



$$\frac{u + V}{2u}$$

probability



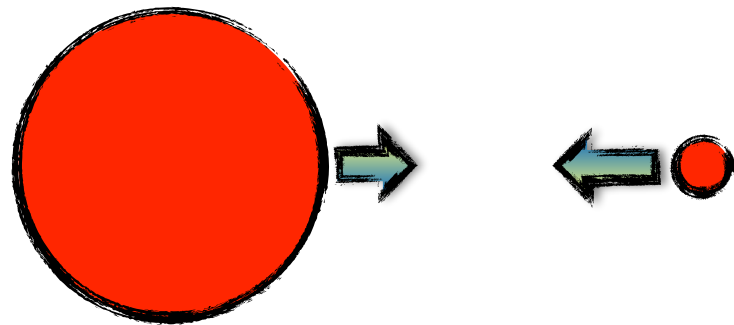
$$\frac{u - V}{2u}$$

$$\Delta E = 2m(V^2 + uV)\frac{u + V}{2u} + 2m(V^2 - uV)\frac{u - V}{2u} = 4mV^2$$

From Fermi's notebook

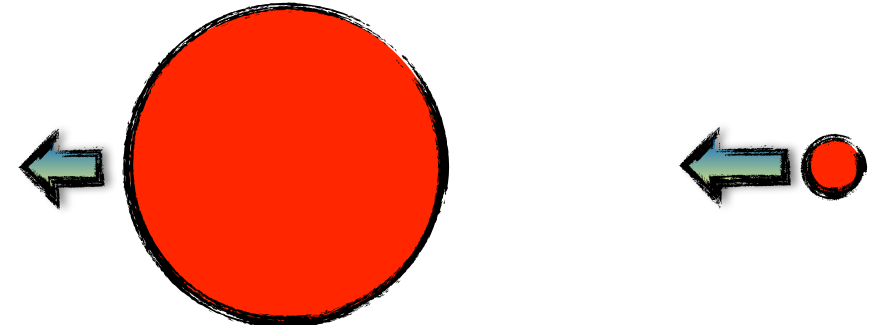
Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

head-on collision -> gain



$$\frac{u + V}{2u}$$

running after collision -> loss



$$\frac{u - V}{2u}$$

probability

$$\Delta E = 2m(V^2 + uV)\frac{u + V}{2u} + 2m(V^2 - uV)\frac{u - V}{2u} = 4mV^2$$

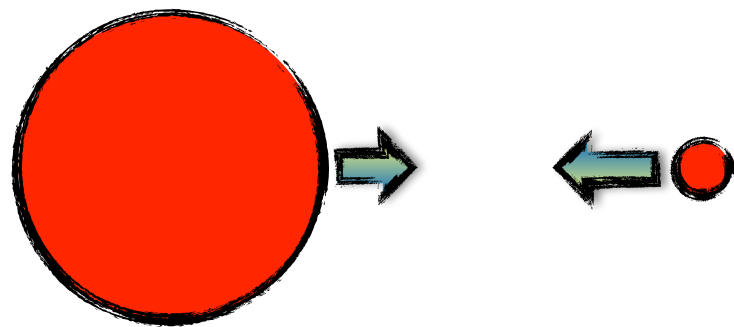
$$\frac{\Delta E}{E} = 8 \left(\frac{V}{u} \right)^2$$

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

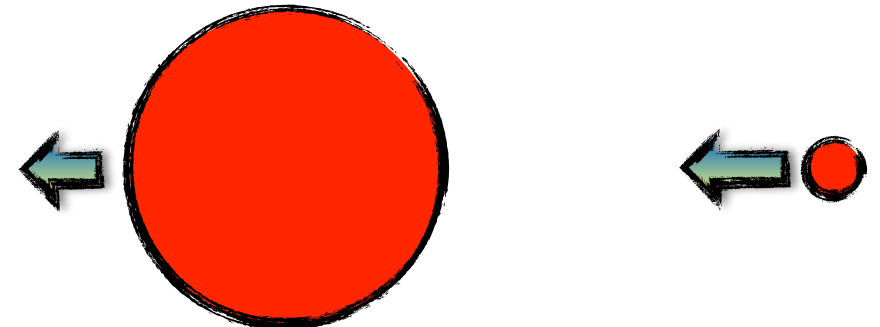
head-on collision -> gain

running after collision -> loss



$$\frac{u + V}{2u}$$

probability



$$\frac{u - V}{2u}$$

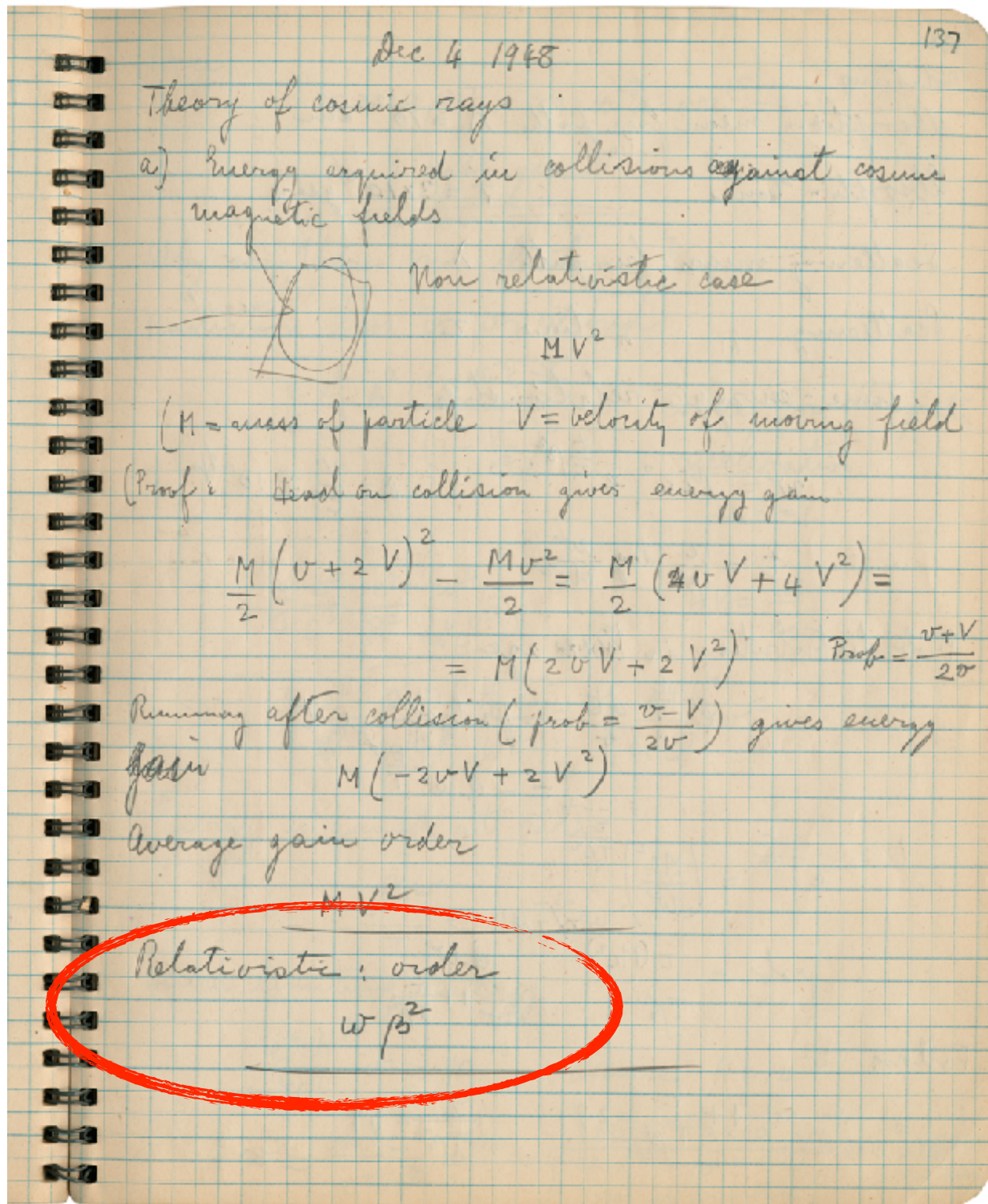
$$\Delta E = 2m(V^2 + uV) \frac{u + V}{2u} + 2m(V^2 - uV) \frac{u - V}{2u} = 4mV^2$$

$$\frac{\Delta E}{E} = 8 \left(\frac{V}{u} \right)^2$$

second order

From Fermi's notebook

Fermi ends his notes saying that for the relativistic case ($u \rightarrow c$) one should expect:

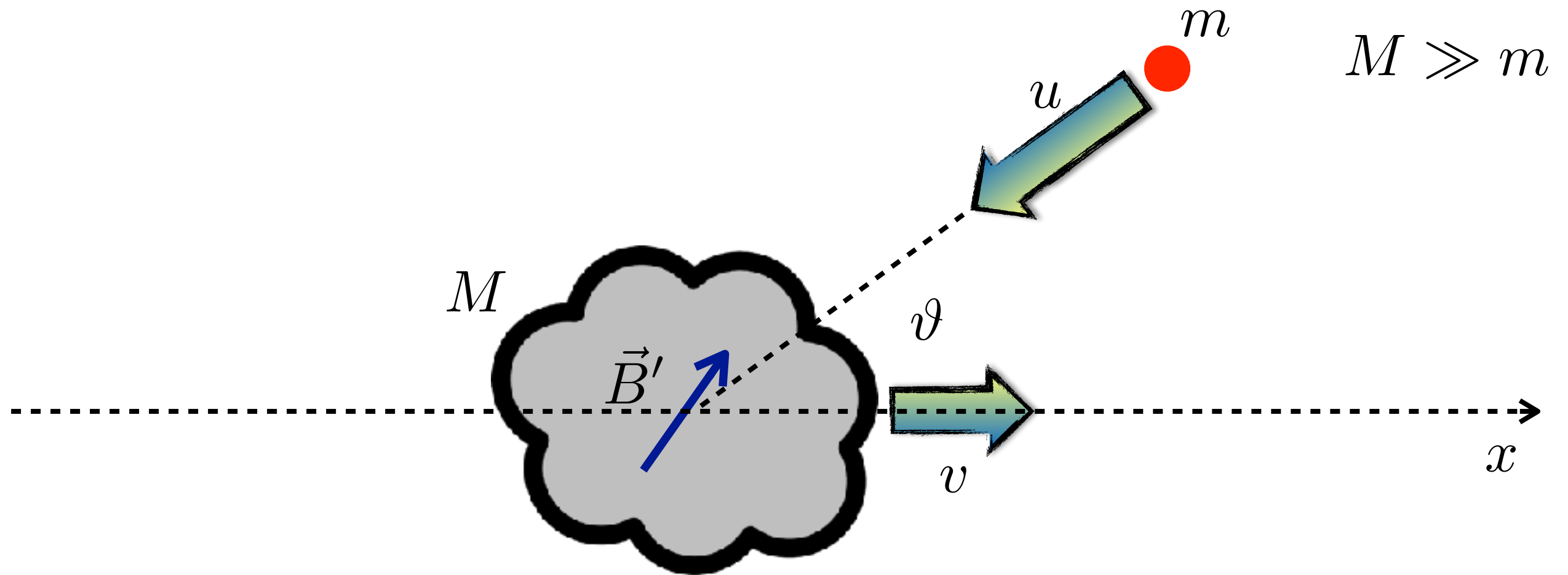


$$\frac{\Delta E}{E} \approx \left(\frac{V}{c} \right)^2 = \beta^2$$

Fermi II

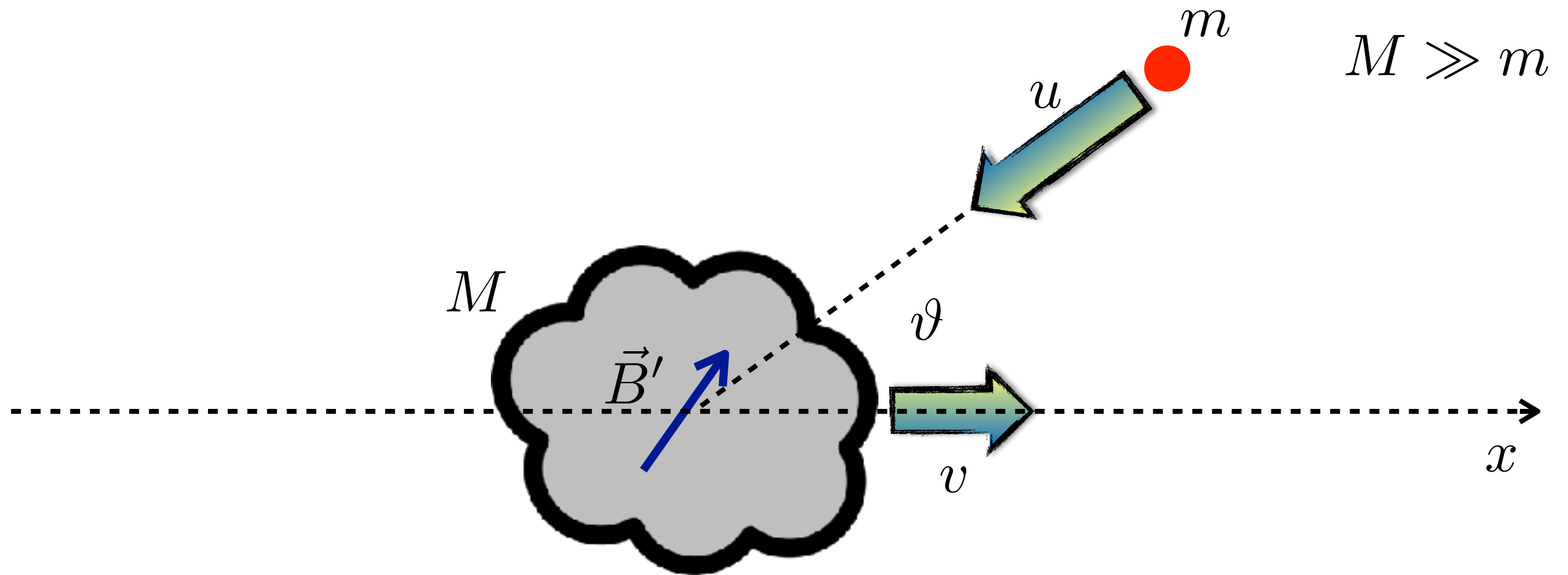
From Fermi's paper

Fermi (1949, 1954)



From Fermi's paper

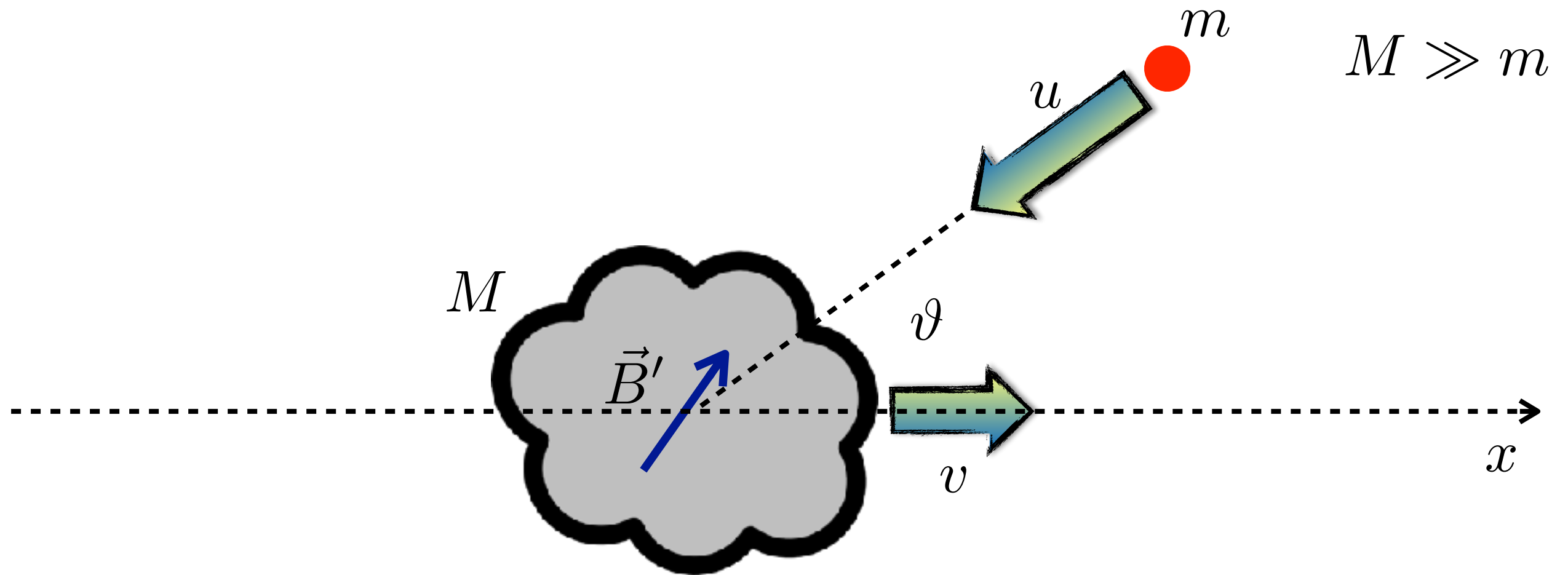
Fermi (1949, 1954)



energy of the particle in the cloud frame $\rightarrow E' = \gamma_v (E + vp \cos \vartheta)$

From Fermi's paper

Fermi (1949, 1954)

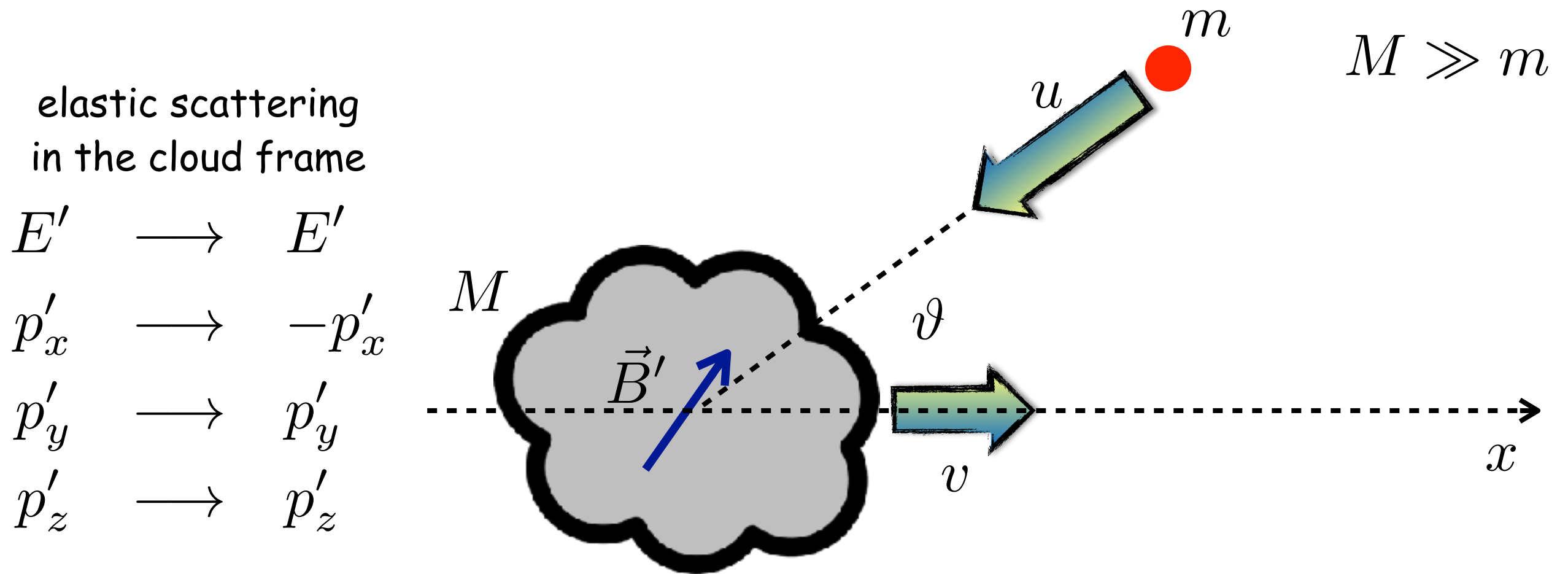


energy of the particle in the cloud frame $\rightarrow E' = \gamma_v (E + vp \cos \vartheta)$

momentum in the cloud frame $\rightarrow p'_x = p' \cos \vartheta' = \gamma_v \left(p \cos \vartheta + \frac{vE}{c^2} \right)$

From Fermi's paper

Fermi (1949, 1954)

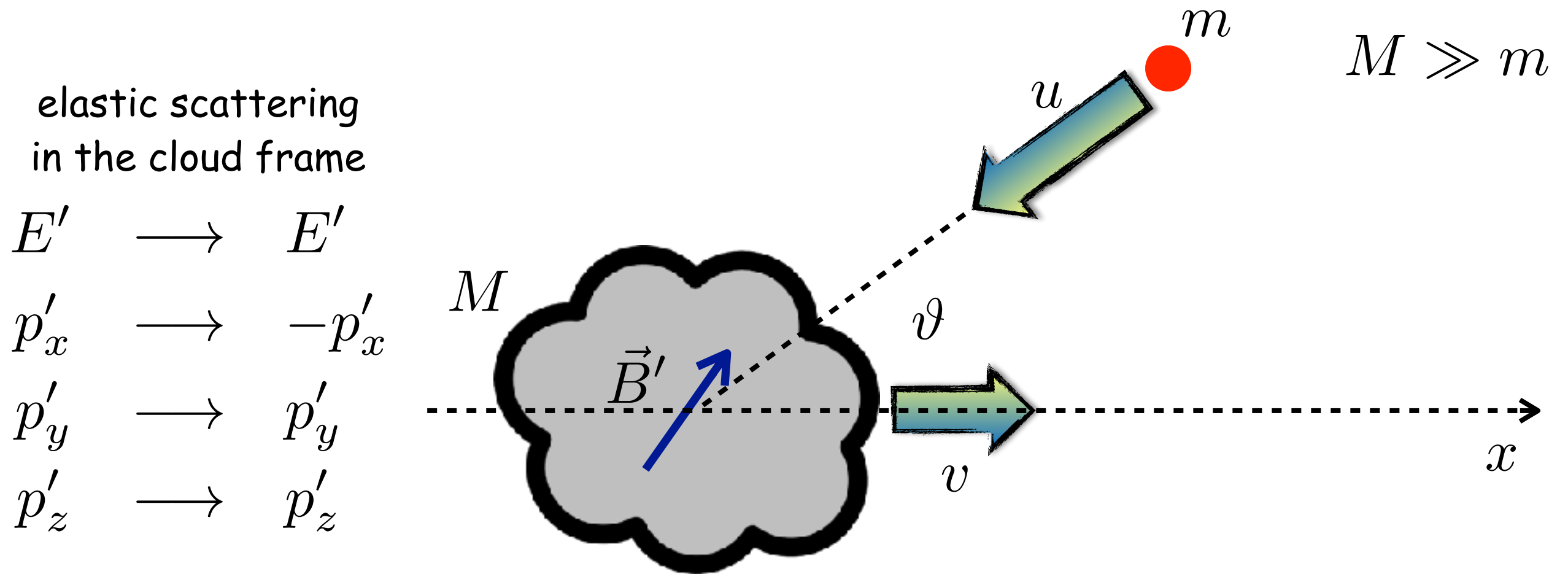


energy of the particle in the cloud frame $\rightarrow E' = \gamma_v (E + vp \cos \vartheta)$

momentum in the cloud frame $\rightarrow p'_x = p' \cos \vartheta' = \gamma_v \left(p \cos \vartheta + \frac{vE}{c^2} \right)$

From Fermi's paper

Fermi (1949, 1954)



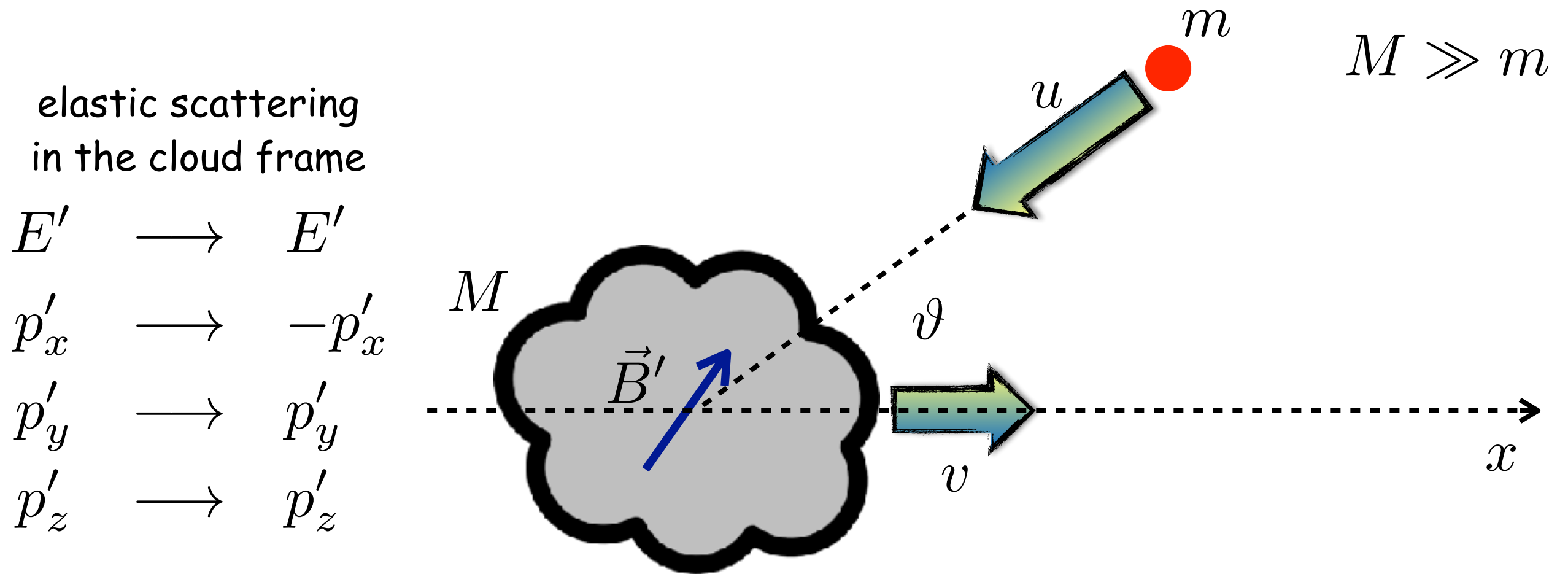
energy of the particle in the cloud frame $\rightarrow E' = \gamma_v (E + vp \cos \vartheta)$

momentum in the cloud frame $\rightarrow p'_x = p' \cos \vartheta' = \gamma_v \left(p \cos \vartheta + \frac{vE}{c^2} \right)$

back to the observer frame $\rightarrow E'' = \gamma_v (E' - vp' \cos \vartheta')$

From Fermi's paper

Fermi (1949, 1954)



energy of the particle in the cloud frame $\rightarrow$ $E' = \gamma_v (E + vp \cos \vartheta)$

momentum in the cloud frame $\rightarrow p'_x = p' \cos \vartheta' = \gamma_v \left(p \cos \vartheta + \frac{vE}{c^2} \right)$

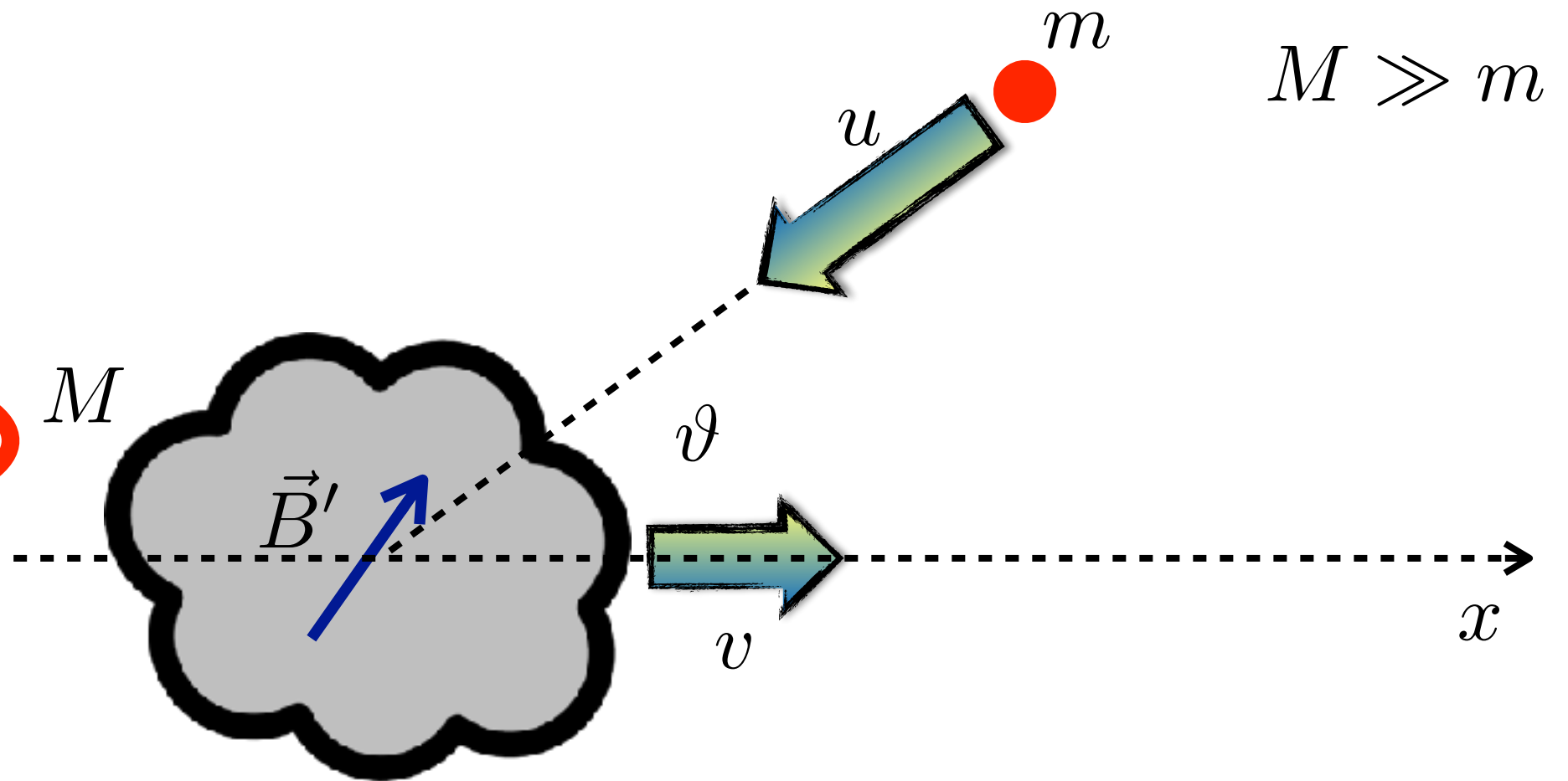
back to the observer frame $\rightarrow E'' = \gamma_v (E' - vp' \cos \vartheta')$

From Fermi's paper

Fermi (1949, 1954)

elastic scattering
in the cloud frame

$$\begin{array}{ccc} E' & \longrightarrow & E' \\ p'_x & \longrightarrow & -p'_x \\ p'_y & \longrightarrow & p'_y \\ p'_z & \longrightarrow & p'_z \end{array}$$



energy of the particle in the cloud frame $\rightarrow$ $E' = \gamma_v (E + vp \cos \vartheta)$

momentum in the cloud frame $\rightarrow p'_x = p' \cos \vartheta' = \gamma_v \left(p \cos \vartheta + \frac{vE}{c^2} \right)$

back to the observer frame $\rightarrow E'' = \gamma_v (E' - vp' \cos \vartheta')$

From Fermi's paper

Fermi (1949, 1954)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

From Fermi's paper

Fermi (1949, 1954)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

the particle gains energy if:

$$\frac{\Delta E}{E} > 0 \longrightarrow \cos \vartheta > -\frac{v}{u}$$

From Fermi's paper

Fermi (1949, 1954)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

the particle gains energy if:

$$\frac{\Delta E}{E} > 0 \longrightarrow \cos \vartheta > -\frac{v}{u}$$

We now need to average over the angle θ . To do so, we need to know at which rate particles hit the cloud for a given arrival direction.

From Fermi's paper

Fermi (1949, 1954)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

the particle gains energy if:

$$\frac{\Delta E}{E} > 0 \longrightarrow \cos \vartheta > -\frac{v}{u}$$

We now need to average over the angle θ . To do so, we need to know at which rate particles hit the cloud for a given arrival direction.

Probability to collide at an angle $\theta \rightarrow$

$$\left[1 + \left(\frac{v}{c} \right) \cos \vartheta \right]$$

From Fermi's paper

Fermi (1949, 1954)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

the particle gains energy if:

$$\frac{\Delta E}{E} > 0 \longrightarrow \cos \vartheta > -\frac{v}{u}$$

We now need to average over the angle θ . To do so, we need to know at which rate particles hit the cloud for a given arrival direction.

Probability to collide at an angle $\theta \rightarrow$

$$\left[1 + \left(\frac{v}{c} \right) \cos \vartheta \right]$$

relative velocity

From Fermi's paper

Fermi (1949, 1954)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

the particle gains energy if:

$$\frac{\Delta E}{E} > 0 \longrightarrow \cos \vartheta > -\frac{v}{u}$$

We now need to average over the angle θ . To do so, we need to know at which rate particles hit the cloud for a given arrival direction.

Probability to collide at an angle $\theta \rightarrow$

Probability to find θ in $[\theta, \theta + d\theta] \rightarrow$

$$\left[1 + \left(\frac{v}{c} \right) \cos \vartheta \right]$$

$$\sin \vartheta d\vartheta = -d \cos \vartheta$$

relative velocity

From Fermi's paper

Fermi (1949, 1954)

$$\cos \vartheta = x \qquad u \longrightarrow c$$

$$\begin{aligned} \left\langle \frac{\Delta E}{E} \right\rangle &= \left(\frac{2v}{c} \right) \frac{\int_{-1}^1 dx \, x \left[1 + \frac{v}{c} x \right]}{\int_{-1}^1 dx \left[1 + \frac{v}{c} x \right]} + 2 \left(\frac{v}{c} \right)^2 = \\ &= \frac{2}{3} \left(\frac{v}{c} \right)^2 + 2 \left(\frac{v}{c} \right)^2 = \frac{8}{3} \left(\frac{v}{c} \right)^2 \end{aligned}$$

Energy gain, second order, as expected in the initial (notebook) estimate!

Slightly different approach

Fermi considered clouds as magnetic mirrors, in other derivations clouds are considered as scattering centres

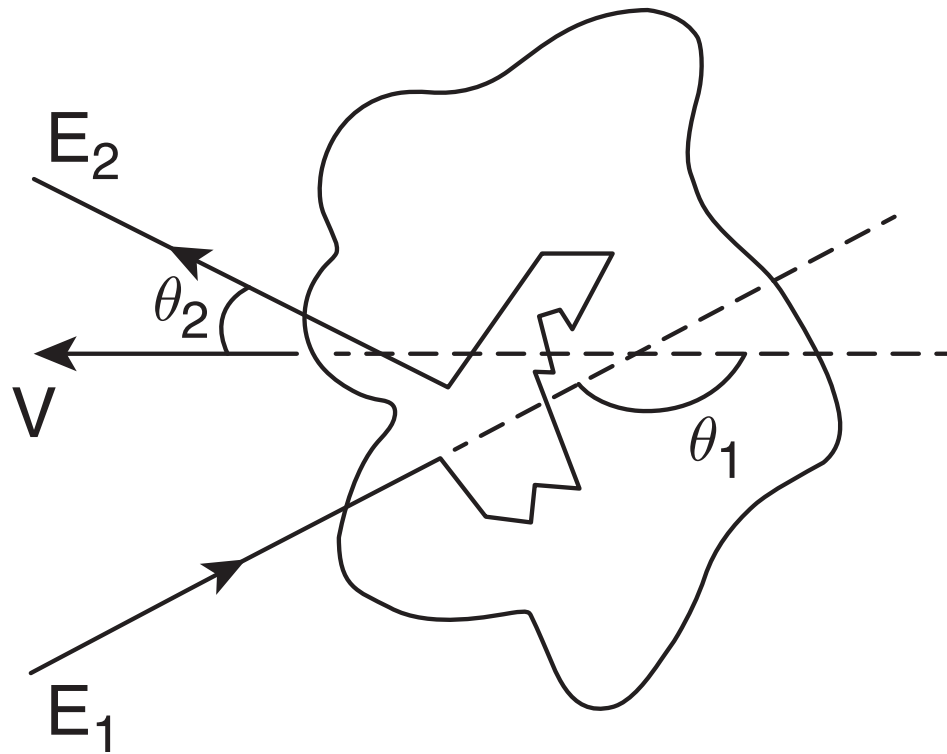
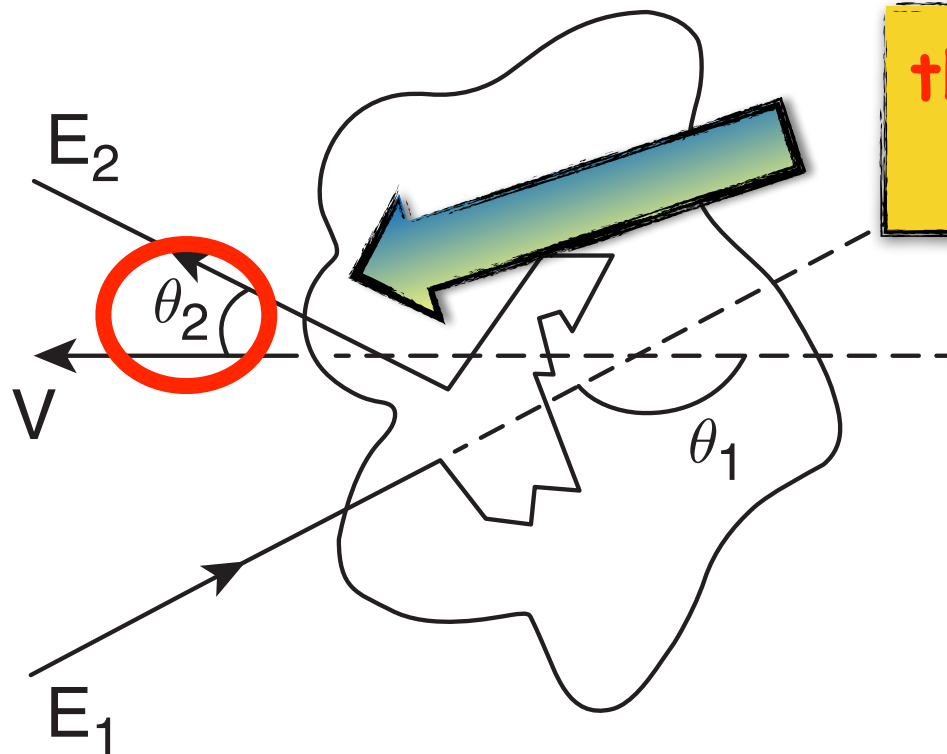


Figure from Gaisser's book

Slightly different approach

Fermi considered clouds as magnetic mirrors, in other derivations clouds are considered as scattering centres

Figure from Gaisser's book



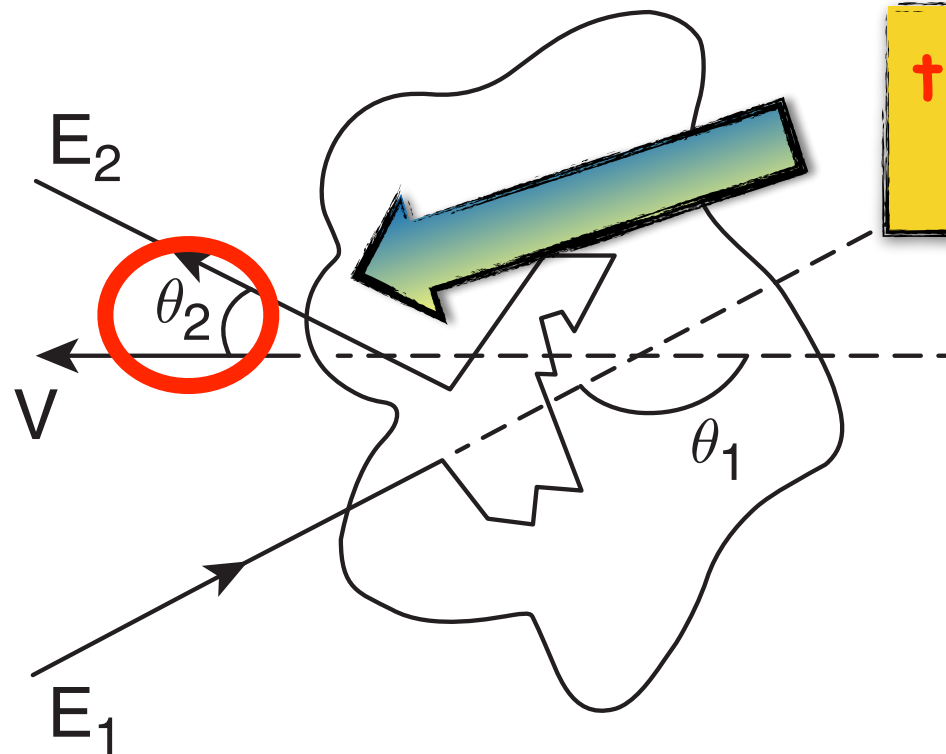
this introduces a dependency on the exit angle

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

Slightly different approach

Fermi considered clouds as magnetic mirrors, in other derivations clouds are considered as scattering centres

Figure from Gaisser's book



this introduces a dependency on the exit angle

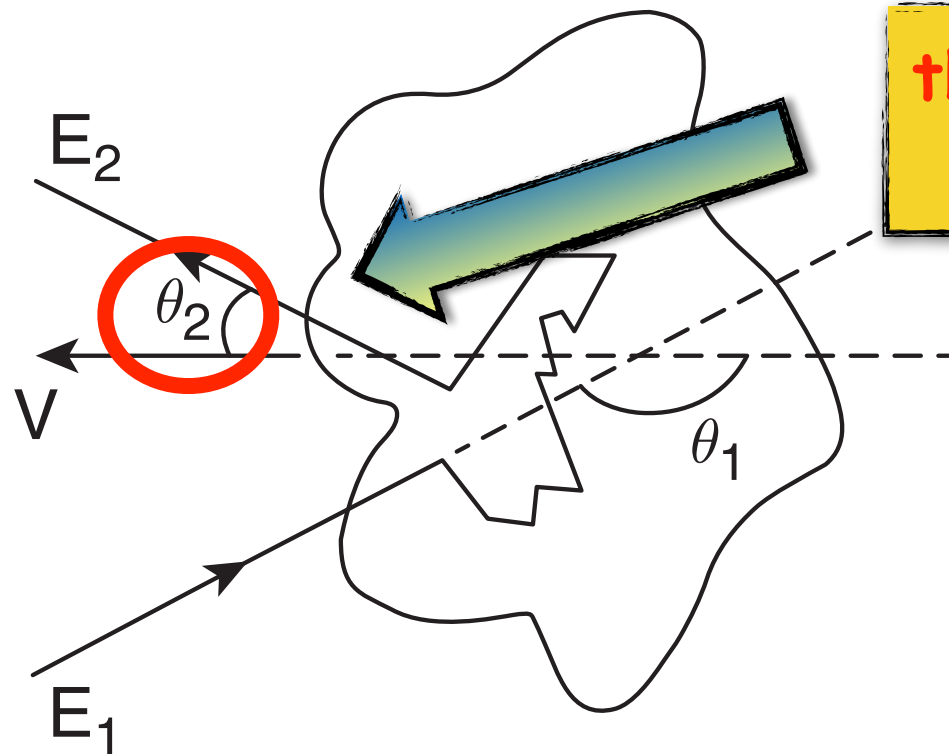
isotropy

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

Slightly different approach

Fermi considered clouds as magnetic mirrors, in other derivations clouds are considered as scattering centres

Figure from Gaisser's book



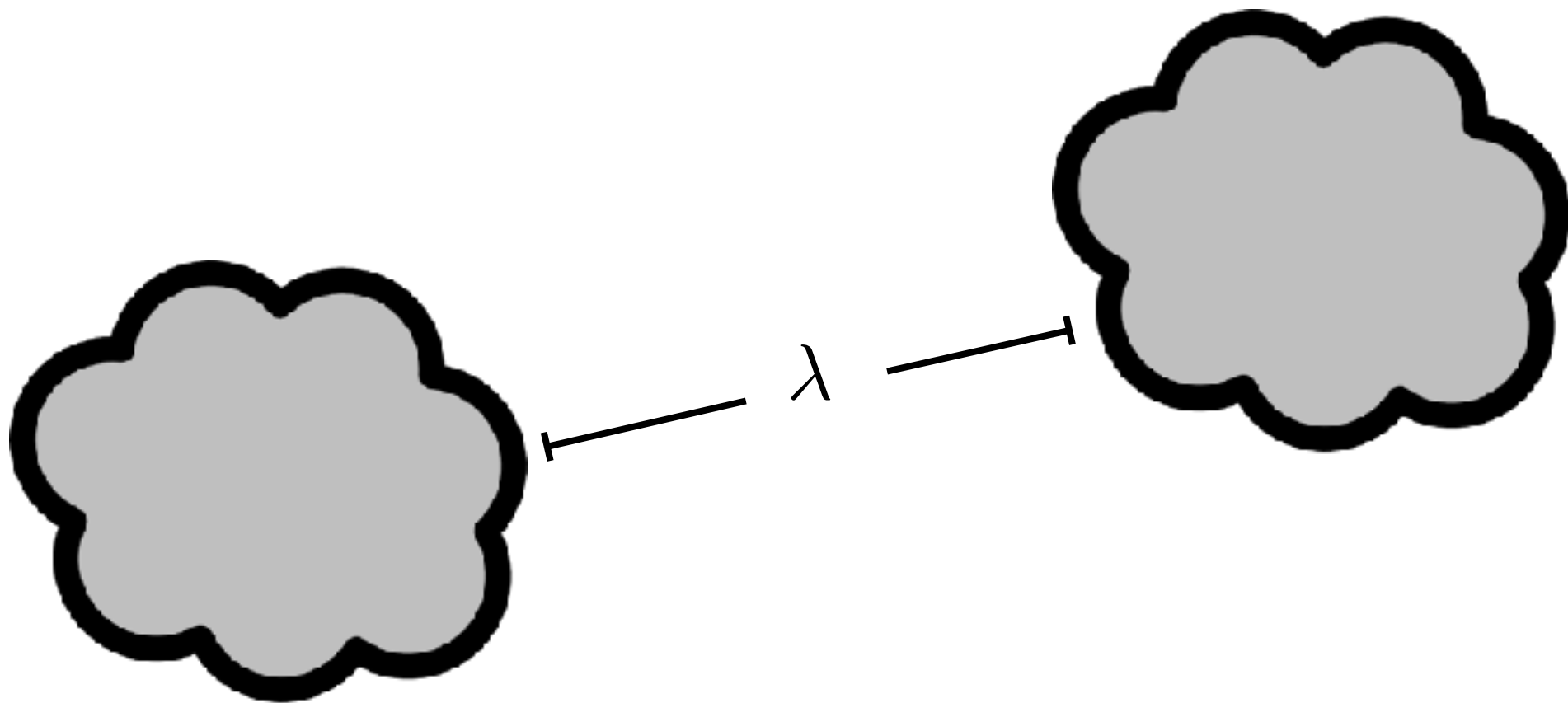
this introduces a dependency on the exit angle

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3} \beta^2$$

isotropy

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

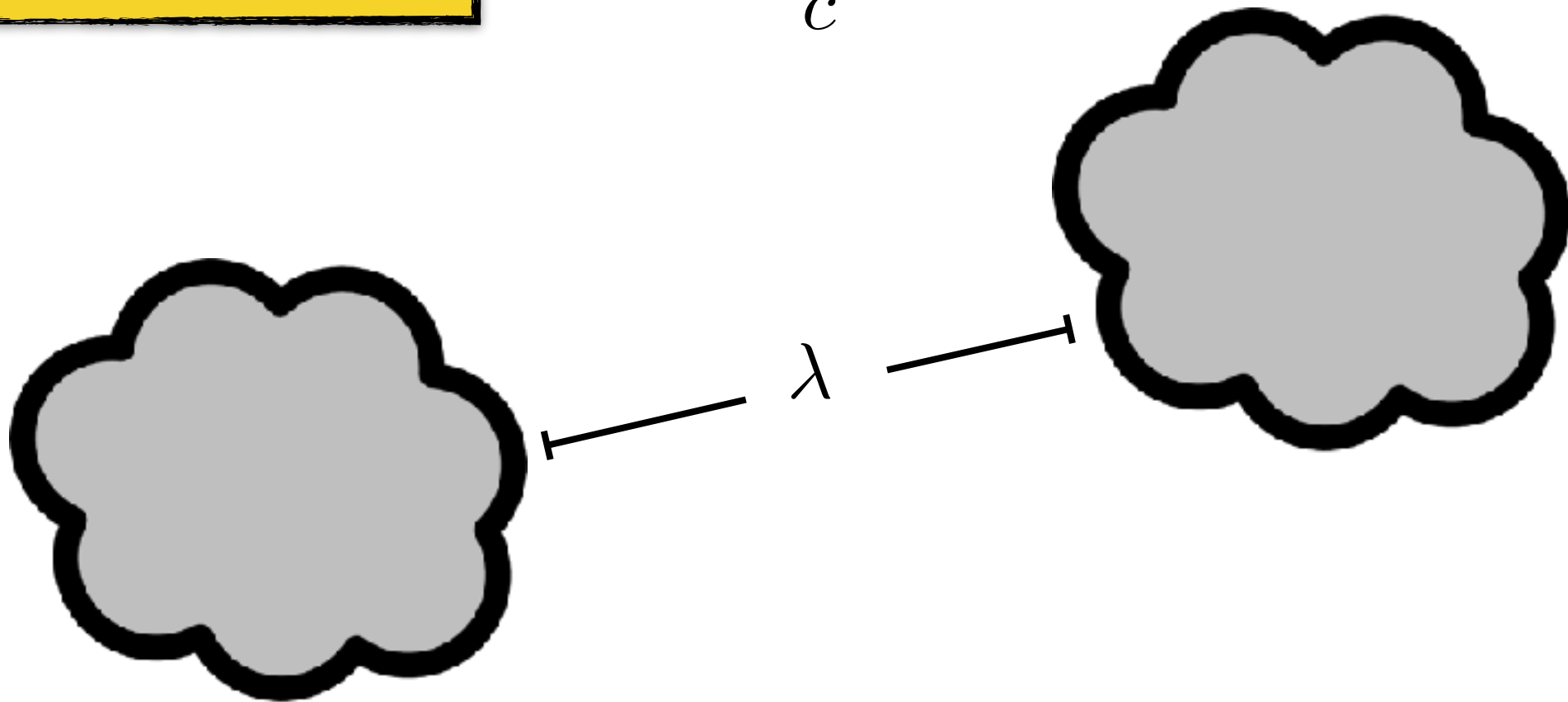
Acceleration rate



Acceleration rate

time between collisions

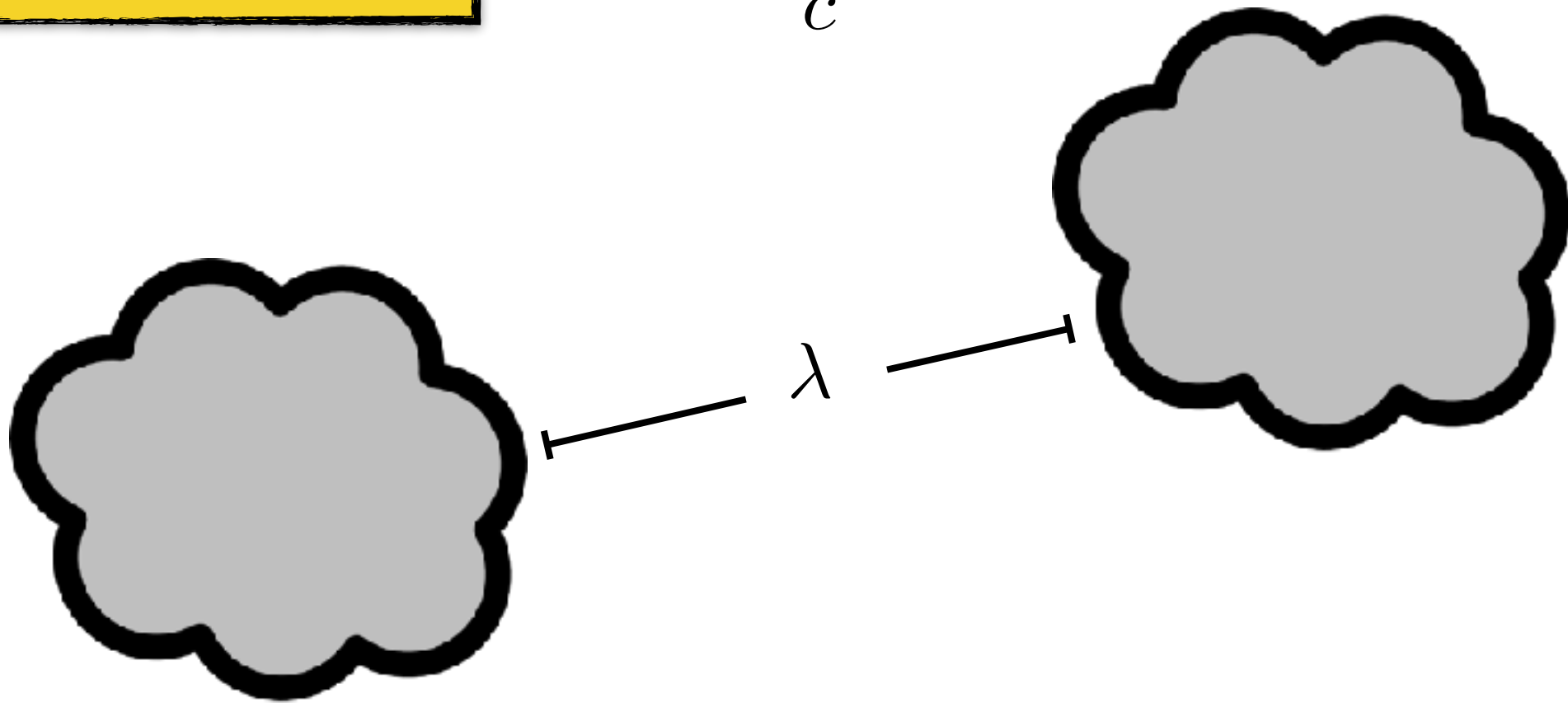
$$\tau_c \approx \frac{\lambda}{c}$$



Acceleration rate

time between collisions

$$\tau_c \approx \frac{\lambda}{c}$$



acceleration rate →

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c} \right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

Good or bad? Compare with Hillas!

Hillas acceleration rate →

Larmor radius

$$\frac{dE}{dt} = qvB = \frac{v}{c} \left(\frac{R_L}{c} \right)^{-1} E$$

Good or bad? Compare with Hillas!

Hillas acceleration rate →

Larmor radius

$$\frac{dE}{dt} = qvB = \frac{v}{c} \left(\frac{R_L}{c} \right)^{-1} E$$

Hillas acceleration time →

$$\tau_{acc}^H = \left(\frac{v}{c} \right)^{-1} \frac{R_L}{c}$$

Good or bad? Compare with Hillas!

Hillas acceleration rate $\rightarrow$

Larmor radius 

$$\frac{dE}{dt} = qvB = \frac{v}{c} \left(\frac{R_L}{c} \right)^{-1} E$$

Hillas acceleration time $\rightarrow$

$$\tau_{acc}^H = \left(\frac{v}{c} \right)^{-1} \frac{R_L}{c}$$

for any acceleration mechanism $\rightarrow$

$$\tau_{acc} = \eta \tau_{acc}^H$$

where of course: $\eta > 1$

Fermi II acceleration time

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-2} \frac{R_L}{c} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1} \tau_{acc}^H$$

energy
independent



$$\eta = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1}$$

Fermi II acceleration time

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-2} \frac{R_L}{c} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1} \tau_{acc}^H$$

energy
independent



order unity



$$\eta = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1}$$

$\gg 1$



$\gg 1$



Fermi II acceleration time

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-2} \frac{R_L}{c} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1} \tau_{acc}^H$$

energy
independent

order unity

$$\eta = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1}$$

$\gg 1$

$\gg 1$

we are very far from the optimal (idealised) Hillas rate

How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

time $\rightarrow t$

$$N(E)\Delta E$$

E

$E + \Delta E$

E

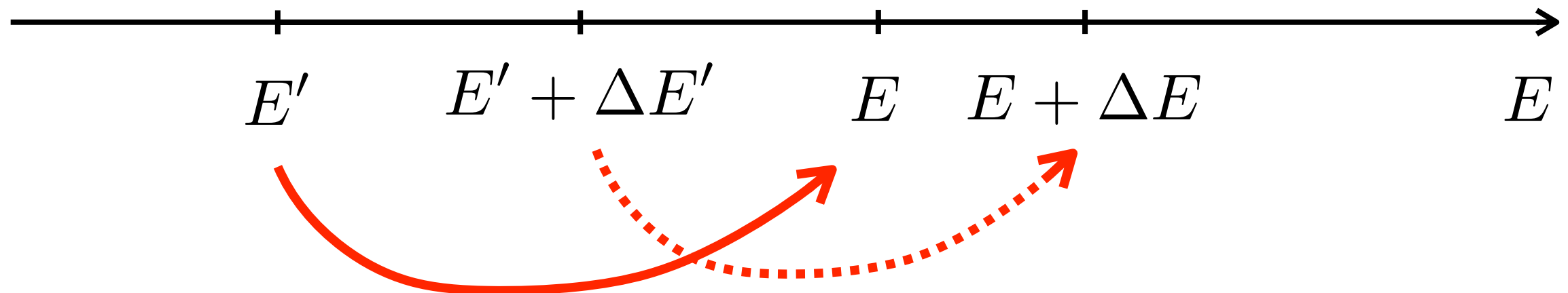


How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

time $\rightarrow t - \Delta t$

$N(E)\Delta E$

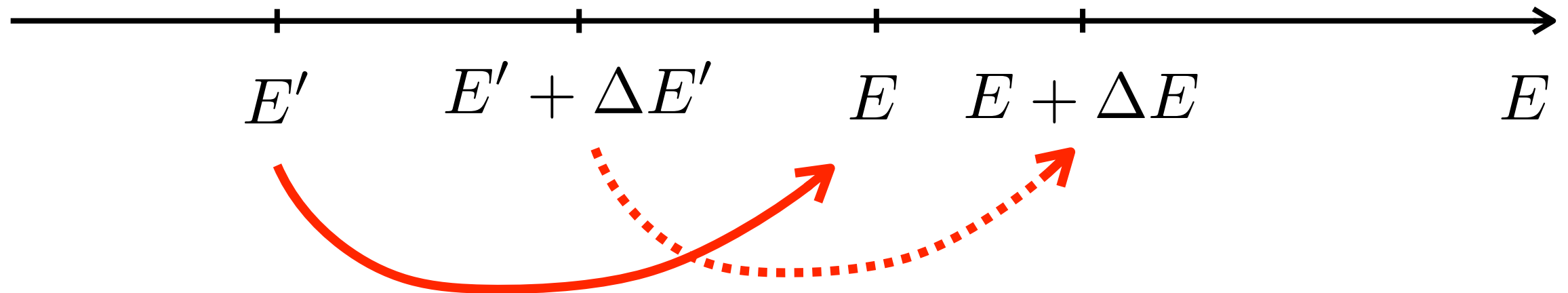


How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

time $\rightarrow t - \Delta t$

$$N(E) \Delta E$$



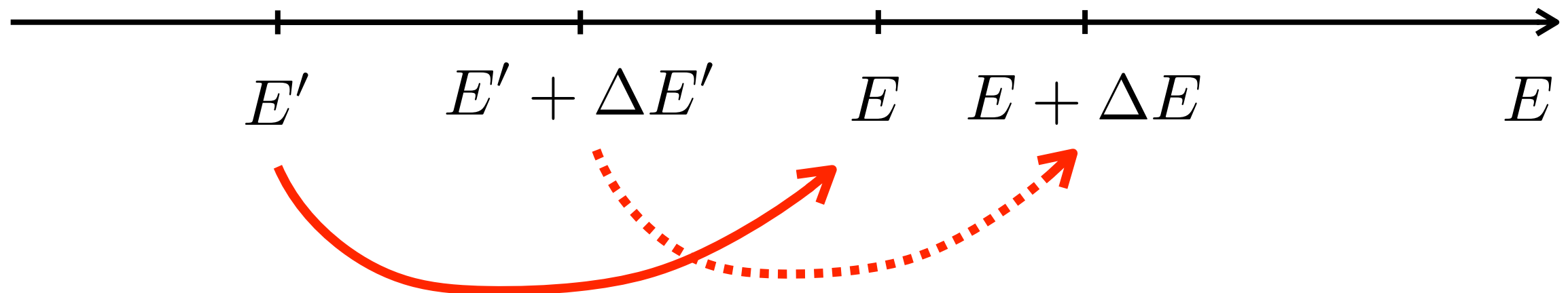
$$E = E' + b(E) \Delta t$$

How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

time $\rightarrow t - \Delta t$

$$N(E) \Delta E$$



$$E = E' + b(E) \Delta t$$

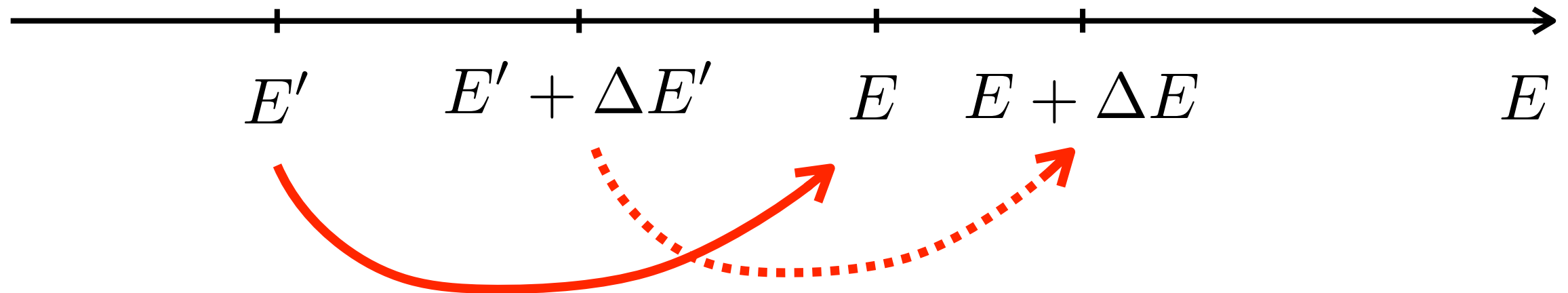
$$E + \Delta E = E' + \Delta E' + b(E + \Delta E) \Delta t$$

How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

time $\rightarrow t - \Delta t$

$$N(E) \Delta E$$



$$E = E' + b(E) \Delta t$$

$$E + \Delta E = E' + \Delta E' + b(E + \Delta E) \Delta t = E' + \Delta E' + b(E) \Delta t + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

Take a generic acceleration rate: $\frac{dE}{dt} = b(E)$

time $\rightarrow t - \Delta t$

$N(E)\Delta E$

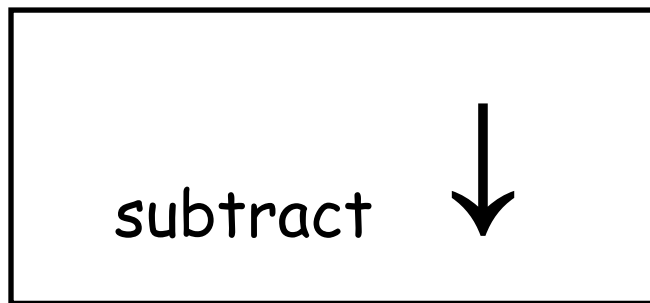
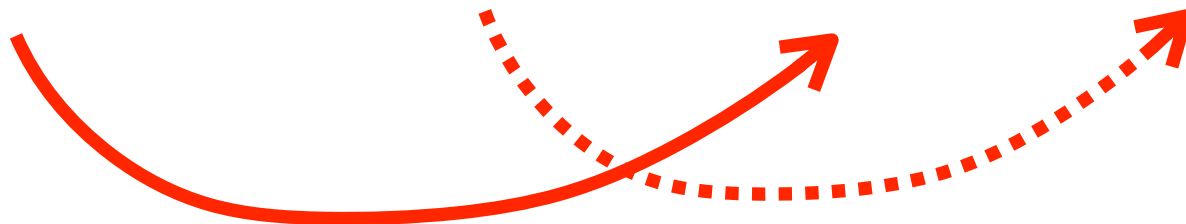
E'

$E' + \Delta E'$

E

$E + \Delta E$

E



$$E = E' + b(E)\Delta t$$

$$E + \Delta E = E' + \text{subtract } \downarrow \Delta t = E' + \Delta E' + b(E)\Delta t + \frac{db(E)}{dE}\Delta E\Delta t$$

$$\Delta E = \Delta E' + \frac{db(E)}{dE}\Delta E\Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \Delta E = N(E', t) \Delta E'$$

$$E = E' + b(E) \Delta t$$
$$\Delta E = \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \Delta E = N(E', t) \Delta E' = N(E - b(E) \Delta t, t) \left[\Delta E - \frac{db(E)}{dE} \Delta E \Delta t \right]$$

$$E = E' + b(E) \Delta t$$

$$\Delta E = \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \cancel{\Delta E} = N(E', t) \Delta E' = N(E - b(E)\Delta t, t) \left[\cancel{\Delta E} - \frac{db(E)}{dE} \cancel{\Delta E} \Delta t \right]$$

$$E = E' + b(E)\Delta t$$

$$\Delta E = \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \cancel{\Delta E} = N(E', t) \Delta E' = N(E - b(E) \Delta t, t) \left[\cancel{\Delta E} - \frac{db(E)}{dE} \cancel{\Delta E} \Delta t \right]$$

↓

$$N(E, t) - \frac{dN(E, t)}{dE} b(E) \Delta t$$

$$E = E' + b(E) \Delta t$$
$$\Delta E = \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \cancel{\Delta E} = N(E', t) \Delta E' = \boxed{N(E - b(E)\Delta t, t)} \left[\cancel{\Delta E} - \frac{db(E)}{dE} \cancel{\Delta E} \Delta t \right]$$

↓

$$N(E, t) - \frac{dN(E, t)}{dE} b(E) \Delta t$$

$$N(E, t + \Delta t) = N(E, t) - \left[N(E, t) \frac{db(E)}{dE} + \frac{dN(E, t)}{dE} b(E) \right] \Delta t$$

$$E = E' + b(E) \Delta t$$

$$\Delta E = \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \cancel{\Delta E} = N(E', t) \Delta E' = \boxed{N(E - b(E)\Delta t, t)} \left[\cancel{\Delta E} - \frac{db(E)}{dE} \cancel{\Delta E} \Delta t \right]$$

↓

$$N(E, t) - \frac{dN(E, t)}{dE} b(E) \Delta t$$

$$\boxed{N(E, t + \Delta t)} = \boxed{N(E, t)} - \left[N(E, t) \frac{db(E)}{dE} + \frac{dN(E, t)}{dE} b(E) \right] \boxed{\Delta t}$$

$$E = E' + b(E) \Delta t$$

$$\Delta E = \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t) \cancel{\Delta E} = N(E', t) \Delta E' = \boxed{N(E - b(E)\Delta t, t)} \left[\cancel{\Delta E} - \frac{db(E)}{dE} \cancel{\Delta E} \Delta t \right]$$

↓

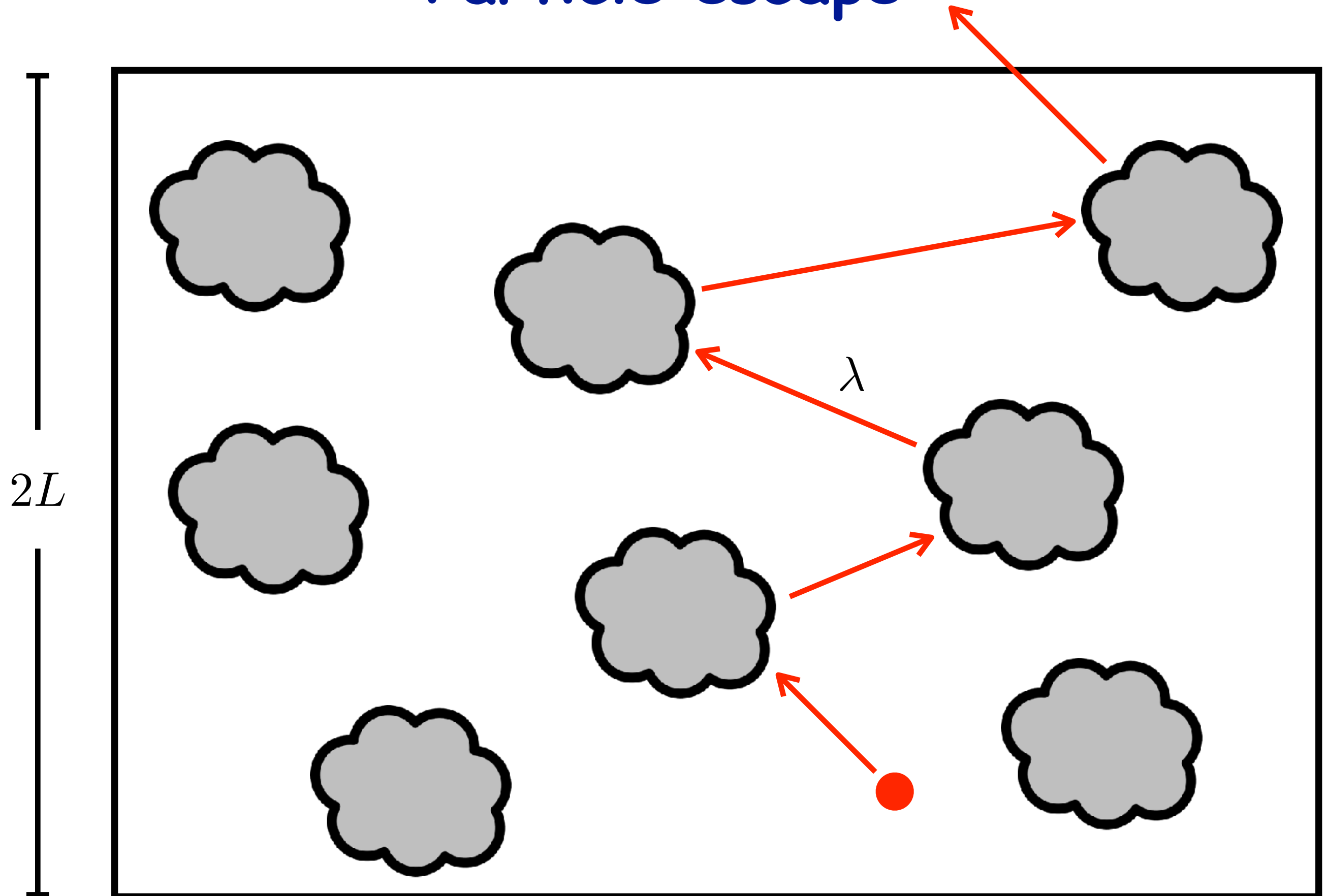
$$N(E, t) - \frac{dN(E, t)}{dE} b(E) \Delta t$$

$$\boxed{N(E, t + \Delta t)} = \boxed{N(E, t)} - \left[N(E, t) \frac{db(E)}{dE} + \frac{dN(E, t)}{dE} b(E) \right] \boxed{\Delta t}$$

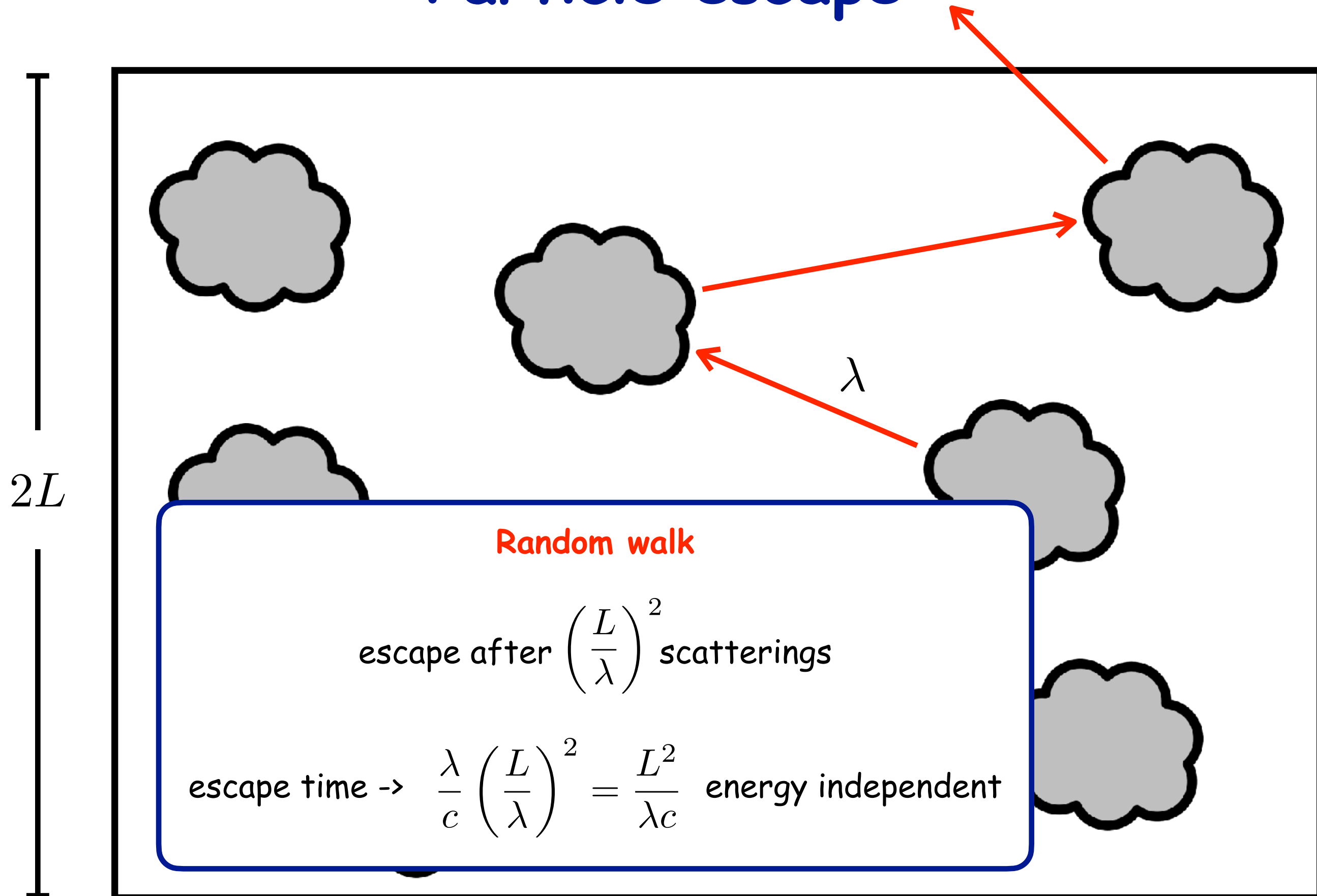
$$\frac{dN(E)}{dt} = - \frac{d}{dE} [b(E) N(E)]$$

$$\begin{aligned} E &= E' + b(E) \Delta t \\ \Delta E &= \Delta E' + \frac{db(E)}{dE} \Delta E \Delta t \end{aligned}$$

Particle escape



Particle escape



Random walk

escape after $\left(\frac{L}{\lambda}\right)^2$ scatterings

$$\text{escape time} \rightarrow \frac{\lambda}{c} \left(\frac{L}{\lambda}\right)^2 = \frac{L^2}{\lambda c} \text{ energy independent}$$

Spectrum of accelerated particles

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}}$$

Spectrum of accelerated particles

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}}$$

Spectrum of accelerated particles

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}}$$

 $\frac{dE}{dt} = \frac{E}{\tau_{acc}}$

Spectrum of accelerated particles

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}}$$

$\frac{dE}{dt} = \frac{E}{\tau_{acc}}$

$$N(E) + E \frac{dN(E)}{dE} = -\frac{\tau_{acc}}{\tau_{esc}} N(E)$$

Spectrum of accelerated particles

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}}$$

$\frac{dE}{dt} = \frac{E}{\tau_{acc}}$

$$N(E) + E \frac{dN(E)}{dE} = -\frac{\tau_{acc}}{\tau_{esc}} N(E)$$

$$\frac{dN(E)}{N(E)} = -\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right) \frac{dE}{E}$$

Spectrum of accelerated particles

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}}$$

$\frac{dE}{dt} = \frac{E}{\tau_{acc}}$

$$N(E) + E \frac{dN(E)}{dE} = -\frac{\tau_{acc}}{\tau_{esc}} N(E)$$

$$\frac{dN(E)}{N(E)} = -\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right) \frac{dE}{E}$$

$$N(E) \propto E^{-\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right)}$$

power law!

Fine tuning problem?

$$N(E) \propto E^{-\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right)}$$

Fine tuning problem?

$$N(E) \propto E^{-\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right)}$$

the ratio $\frac{\tau_{acc}}{\tau_{esc}}$ can be tuned to get ANY spectral slope

with this respect, the model is not very predictive

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c} \right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c} \right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

$$\frac{dE}{E} = \frac{\alpha v^2}{\lambda c} dt \longrightarrow E = E_0 \exp \left[\frac{\alpha v^2}{\lambda c} t \right]$$

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c} \right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

$$\frac{dE}{E} = \frac{\alpha v^2}{\lambda c} dt \longrightarrow E = E_0 \exp \left[\frac{\alpha v^2}{\lambda c} t \right]$$

τ_{acc}^{-1} 

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c} \right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

$$\frac{dE}{E} = \frac{\alpha v^2}{\lambda c} dt \longrightarrow E = E_0 \exp \left[\frac{\alpha v^2}{\lambda c} t \right]$$

τ_{acc}^{-1} 

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} \sim 3 \left(\frac{\lambda}{\text{pc}} \right) \left(\frac{v}{10 \text{ km/s}} \right)^{-2} \text{Gyr}$$

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c}\right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

$$\frac{dE}{E} = \frac{\alpha v^2}{\lambda c} dt \longrightarrow E = E_0 \exp \left[\frac{\alpha v^2}{\lambda c} t \right]$$

τ_{acc}^{-1} 

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} \sim 3 \left(\frac{\lambda}{\text{pc}} \right) \left(\frac{v}{10 \text{ km/s}} \right)^{-2} \text{Gyr}$$

cosmic rays remain confined within the Galaxy for ~10-20 million years, and then they escape in intergalactic space...

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c}\right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

$$\frac{dE}{E} = \frac{\alpha v^2}{\lambda c} dt \longrightarrow E = E_0 \exp \left[\frac{\alpha v^2}{\lambda c} t \right]$$

τ_{acc}^{-1} 

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} \sim 3 \left(\frac{\lambda}{\text{pc}} \right) \left(\frac{v}{10 \text{ km/s}} \right)^{-2} \text{Gyr}$$

cosmic rays remain confined within the Galaxy for ~10-20 million years, and then they escape in intergalactic space...

too long!!!

Problem #1: too slow

acceleration rate $\rightarrow$

$$\frac{dE}{dt} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c} \right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

$$\frac{dE}{E} = \frac{\alpha v^2}{\lambda c} dt \longrightarrow E = E_0 \exp \left[\frac{\alpha v^2}{\lambda c} t \right]$$

τ_{acc}^{-1} 

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} \sim 3 \left(\frac{\lambda}{\text{pc}} \right) \left(\frac{v}{10 \text{ km/s}} \right)^{-2} \text{Gyr}$$

cosmic rays remain confined within the Galaxy for ~10-20 million years, and then they escape in intergalactic space...

too long!!!

brutal approach: in real life this is important up to ~GeV energies

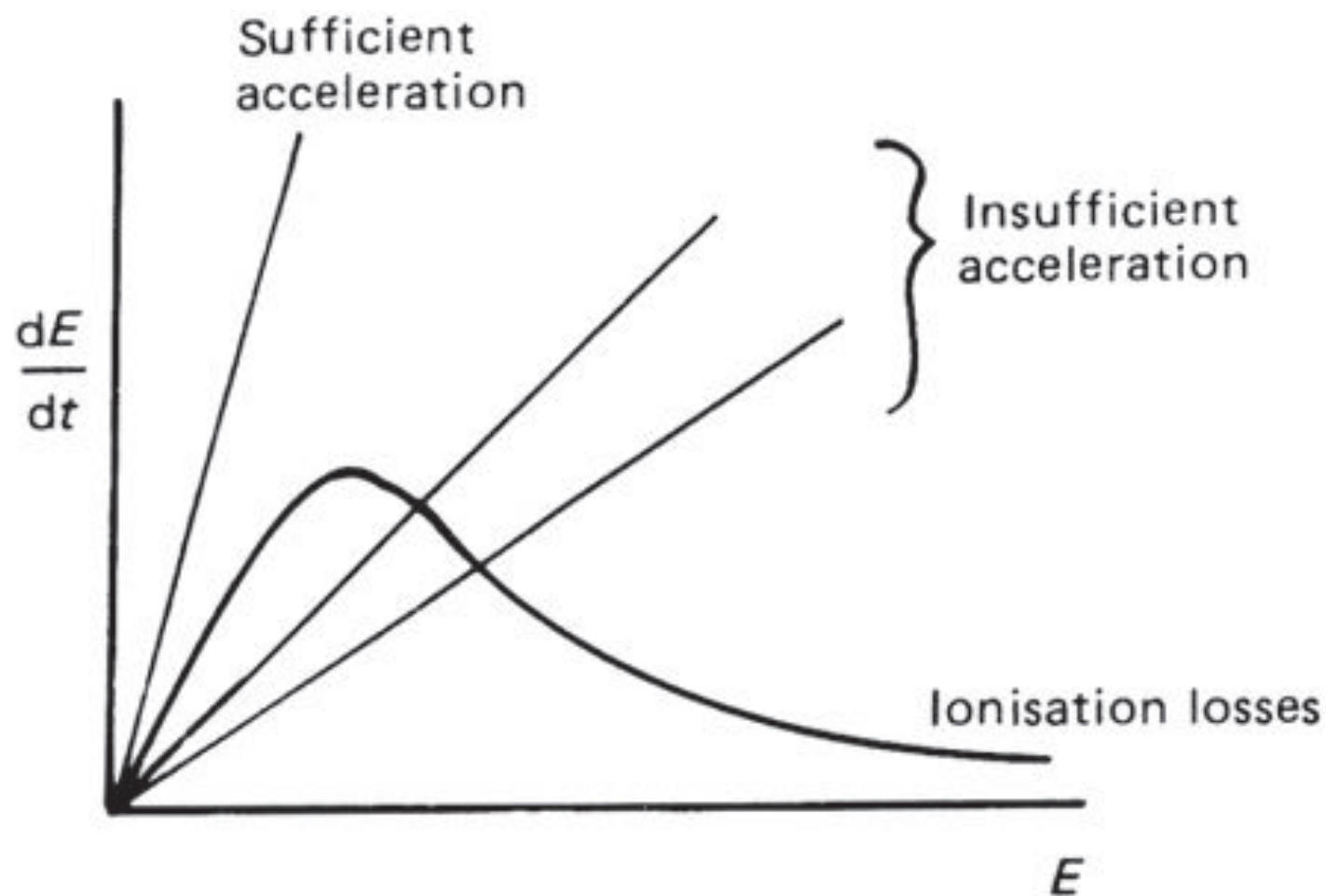
Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

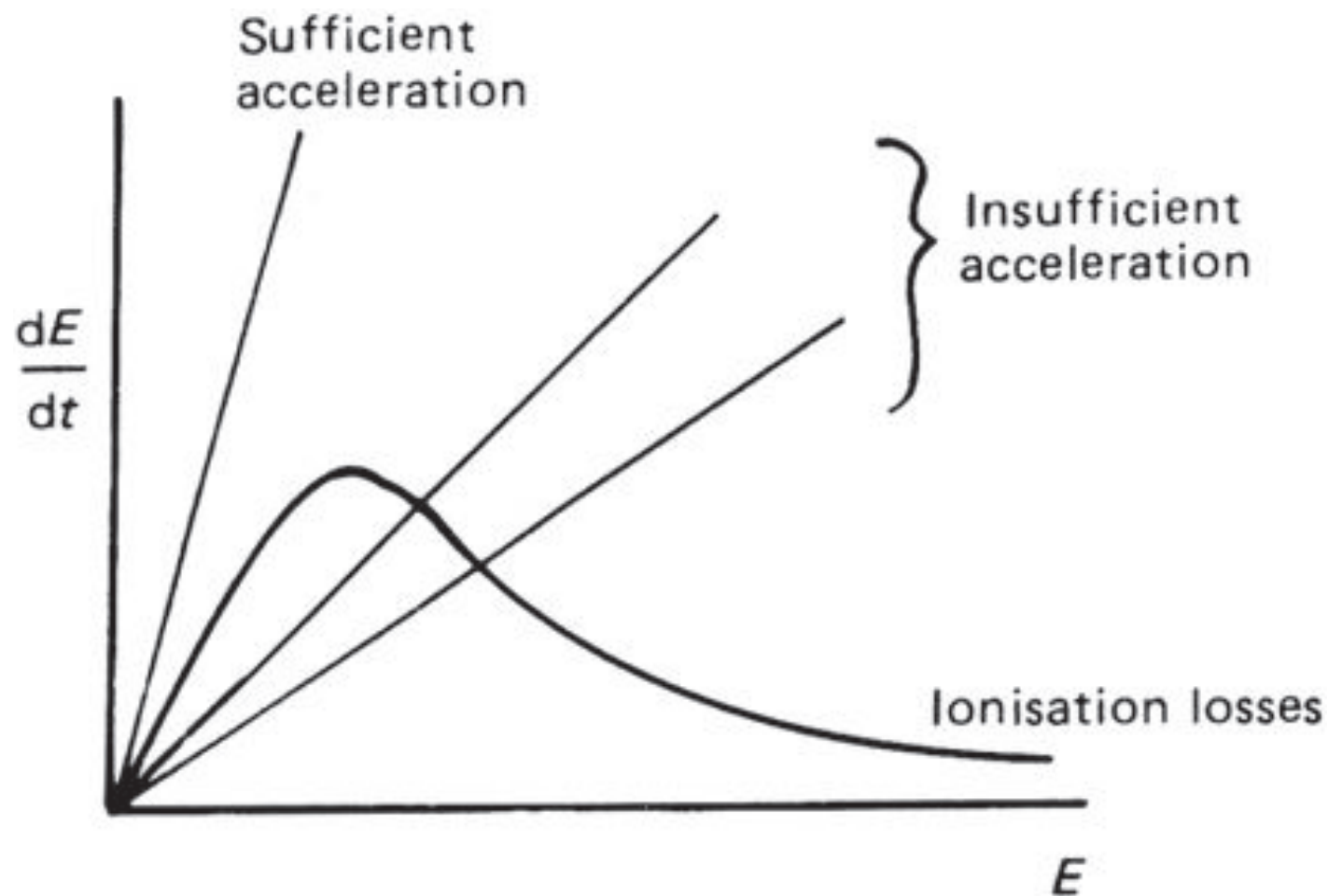
Figure from Longair's book



Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

Figure from Longair's book

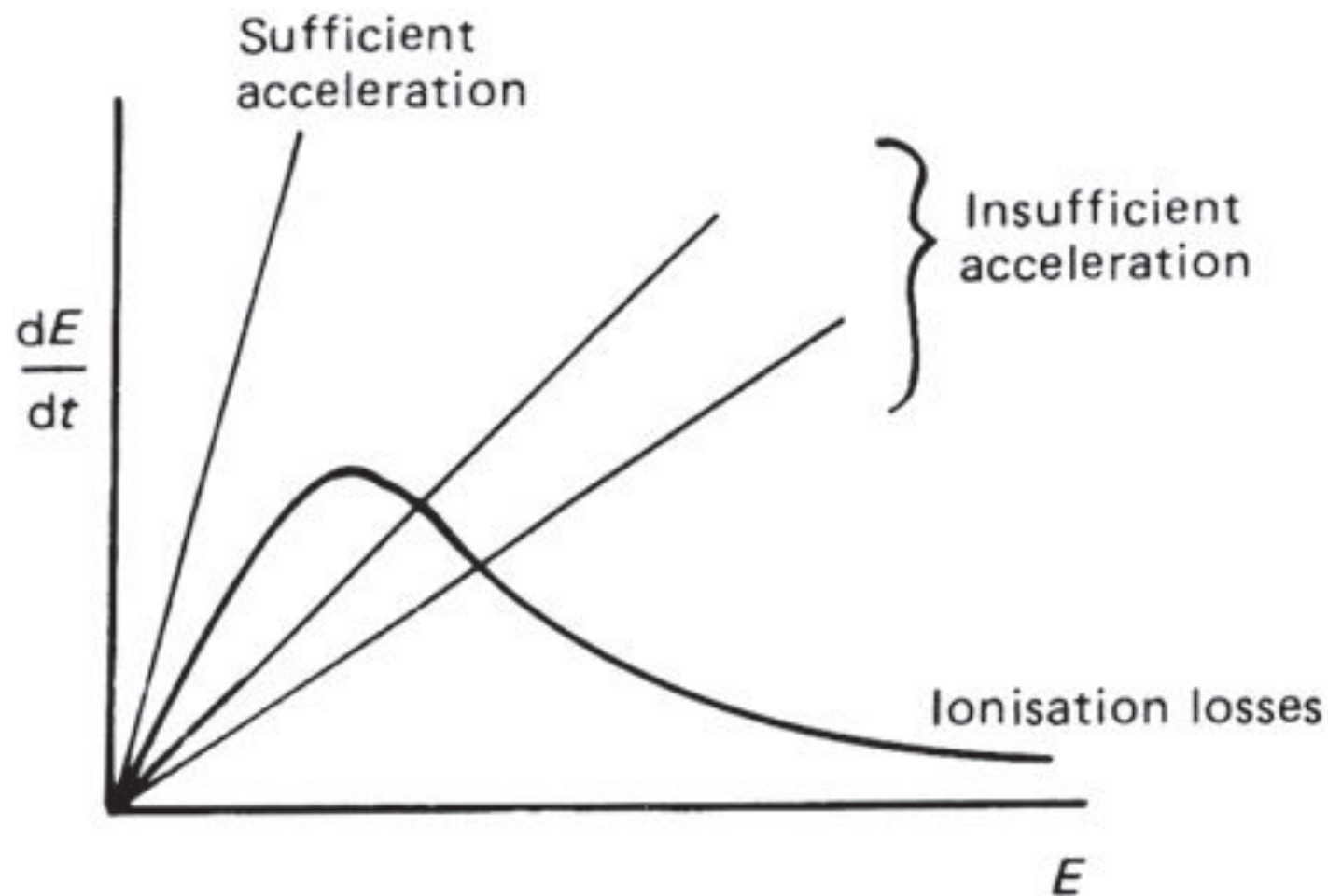


Two ways to overcome losses

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

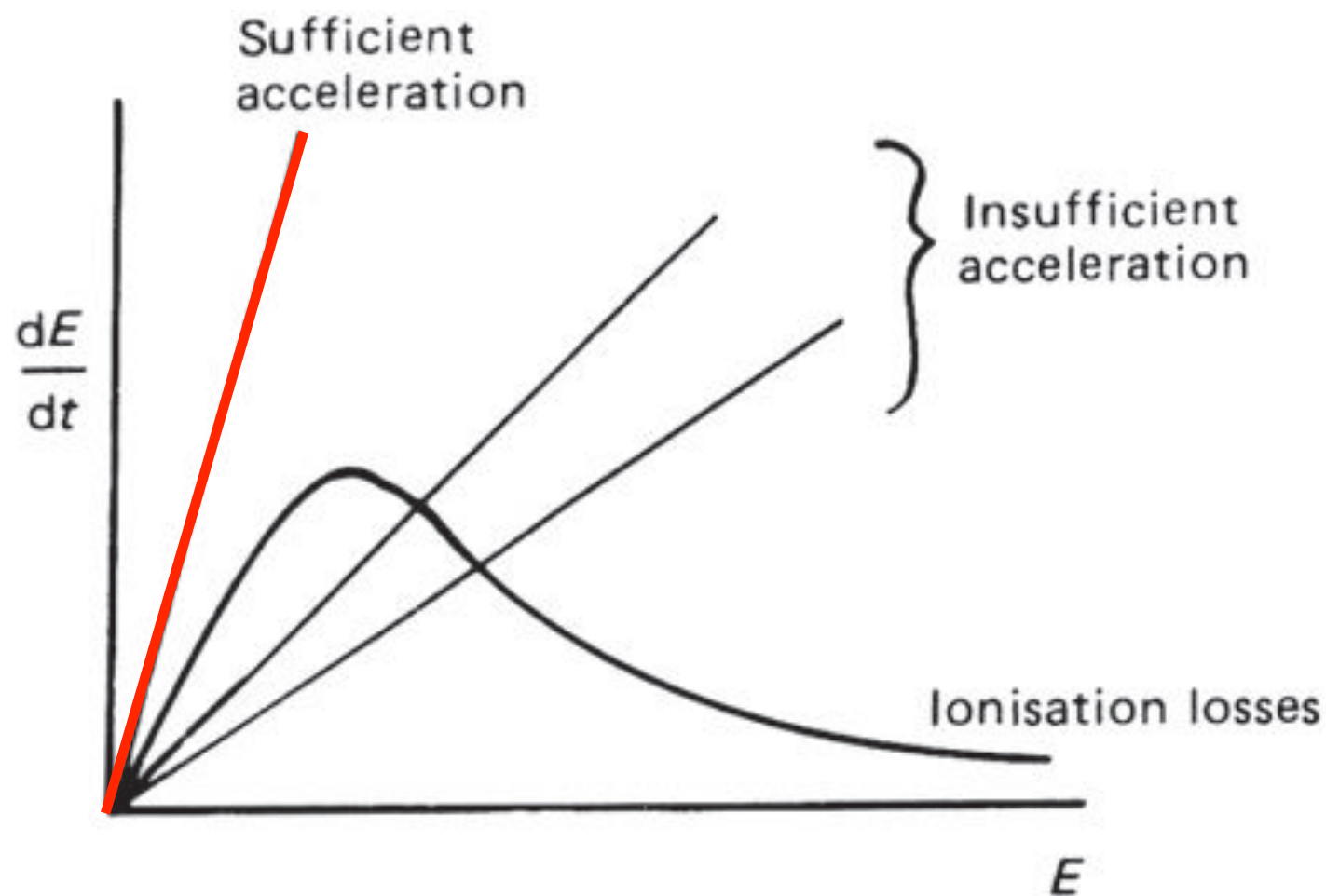
Figure from Longair's book



Two ways to overcome losses

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

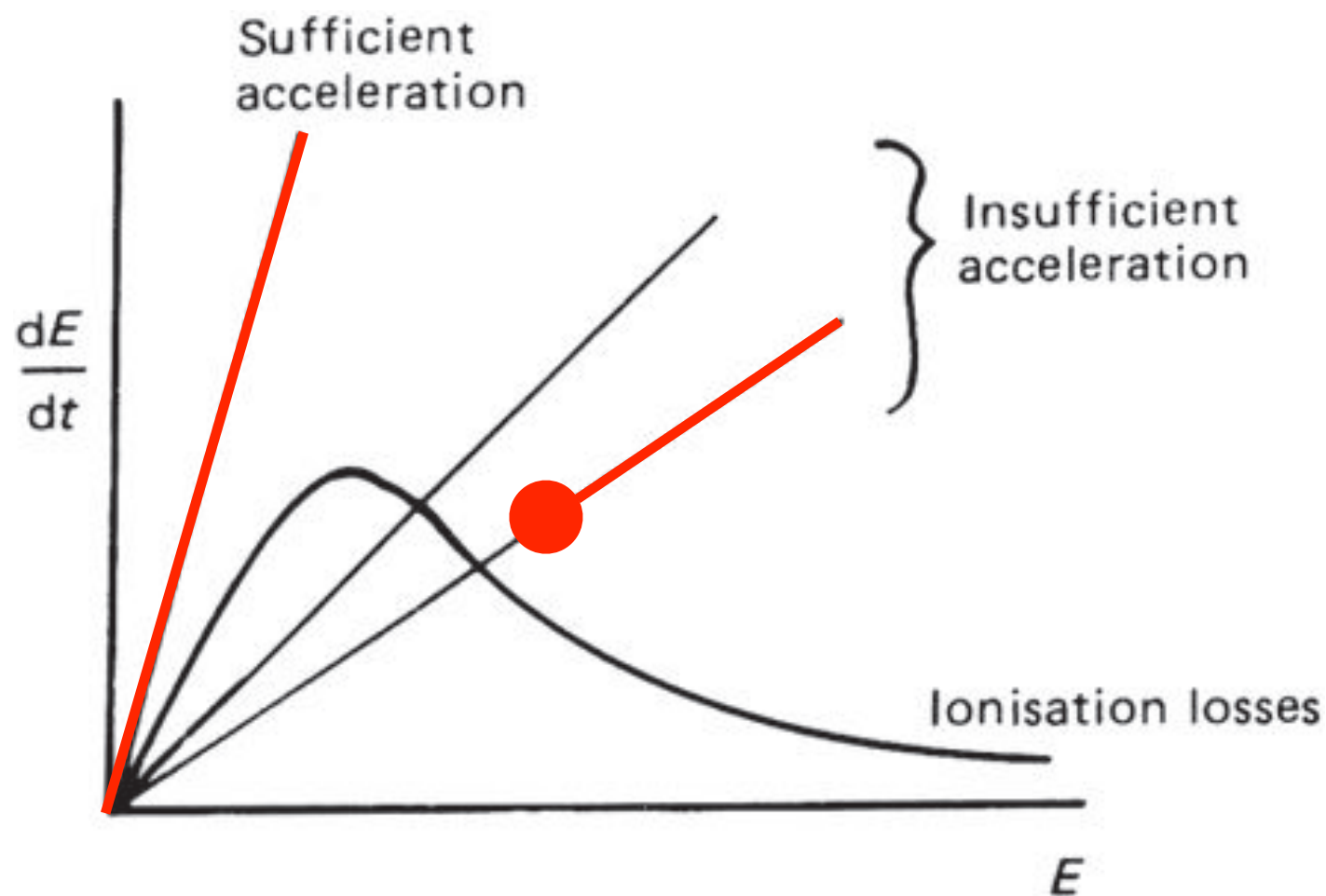


Two ways to overcome losses

■ fast acceleration rate

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses



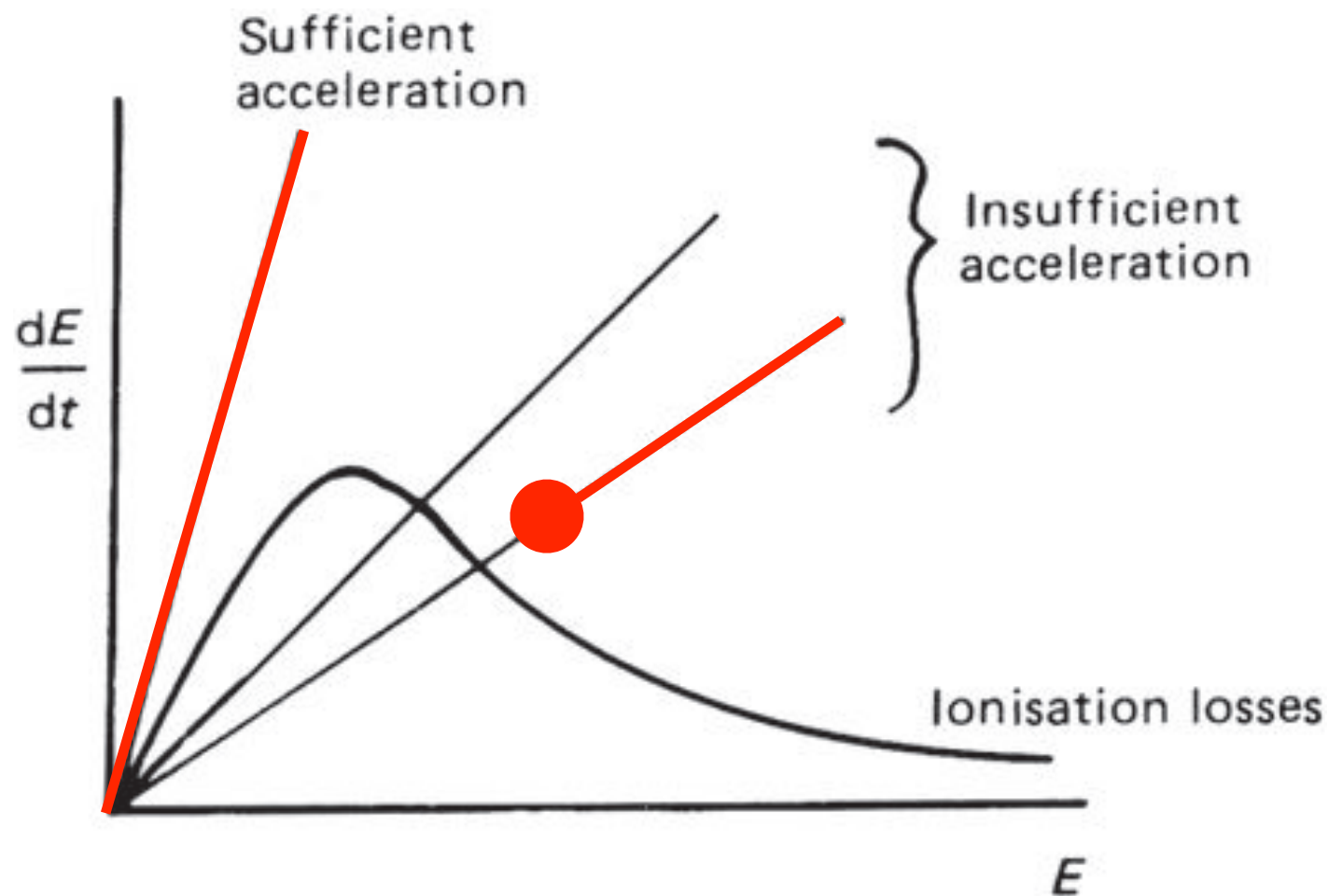
Two ways to overcome losses

- fast acceleration rate
- some process should inject particles at energies large enough to overcome losses

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

Figure from Longair's book



Two ways to overcome losses

- fast acceleration rate
- some process should inject particles at energies large enough to overcome losses

this is known as "injection problem" and belongs to all acceleration mechanisms

Root mean square change in energy

energy gain/loss in a interaction:

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

Root mean square change in energy

energy gain/loss in a interaction:

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2\frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

$$(\Delta E)^2 = 4E^2 \left(\frac{v}{c} \right)^2 \left[\left(\frac{u}{c} \right)^2 \cos^2 \vartheta + \left(\frac{v}{c} \right)^2 + 2\frac{uv}{c^2} \cos \vartheta \right]$$

Root mean square change in energy

energy gain/loss in a interaction:

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2\frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

$$(\Delta E)^2 = 4E^2 \left(\frac{v}{c} \right)^2 \left[\left(\frac{u}{c} \right)^2 \cos^2 \vartheta + \left(\frac{v}{c} \right)^2 + 2\frac{uv}{c^2} \cos \vartheta \right]$$

$u \longrightarrow c$

keep only second order terms in v/c

$$(\Delta E)^2 = 4E^2 \left(\frac{v}{c} \right)^2 \cos^2 \vartheta$$

Root mean square change in energy

energy gain/loss in a interaction:

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2 \frac{v}{c} \left[\frac{u}{c} \cos \vartheta + \frac{v}{c} \right]$$

$$(\Delta E)^2 = 4E^2 \left(\frac{v}{c} \right)^2 \left[\left(\frac{u}{c} \right)^2 \cos^2 \vartheta + \left(\frac{v}{c} \right)^2 + 2 \frac{uv}{c^2} \cos \vartheta \right]$$

$u \longrightarrow c$

keep only second order terms in v/c

$$(\Delta E)^2 = 4E^2 \left(\frac{v}{c} \right)^2 \cos^2 \vartheta$$

average over angles as already did to derive Fermi's result

$$\langle (\Delta E)^2 \rangle = \frac{4}{3} E^2 \left(\frac{v}{c} \right)^2$$

Systematic versus root mean square change in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

Systematic versus root mean square change in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

Systematic versus root mean square change in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

Systematic versus root mean square change in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

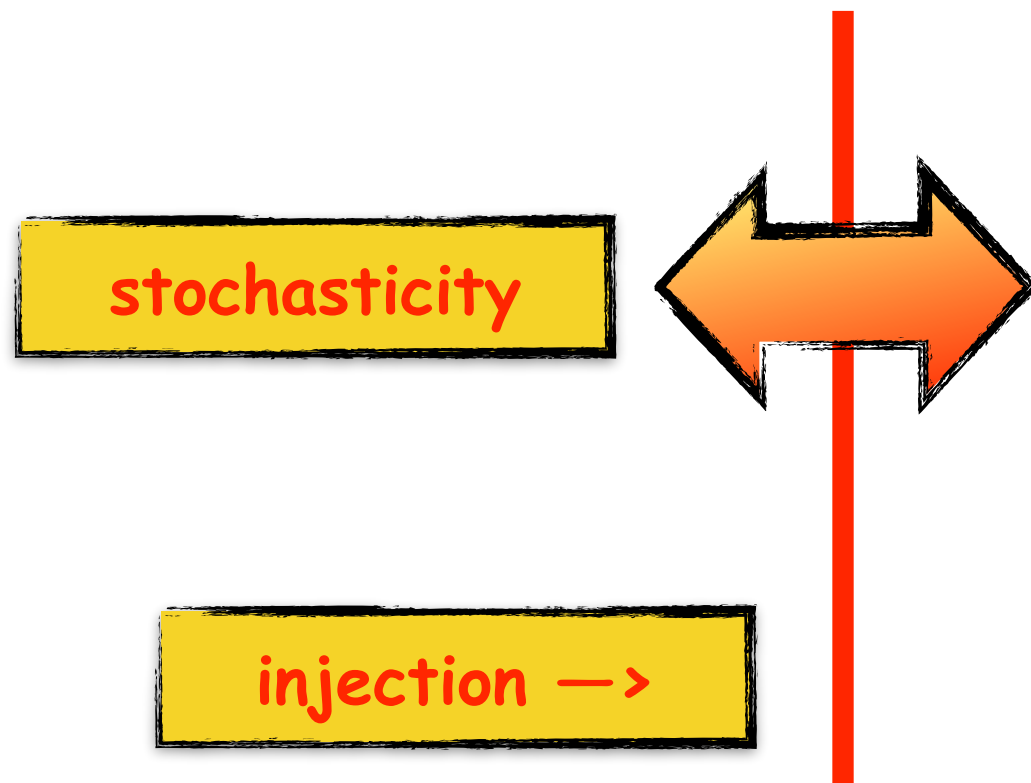
the transport equation we just derived takes into account only the systematic change in particle energy

Systematic versus root mean square change in energy

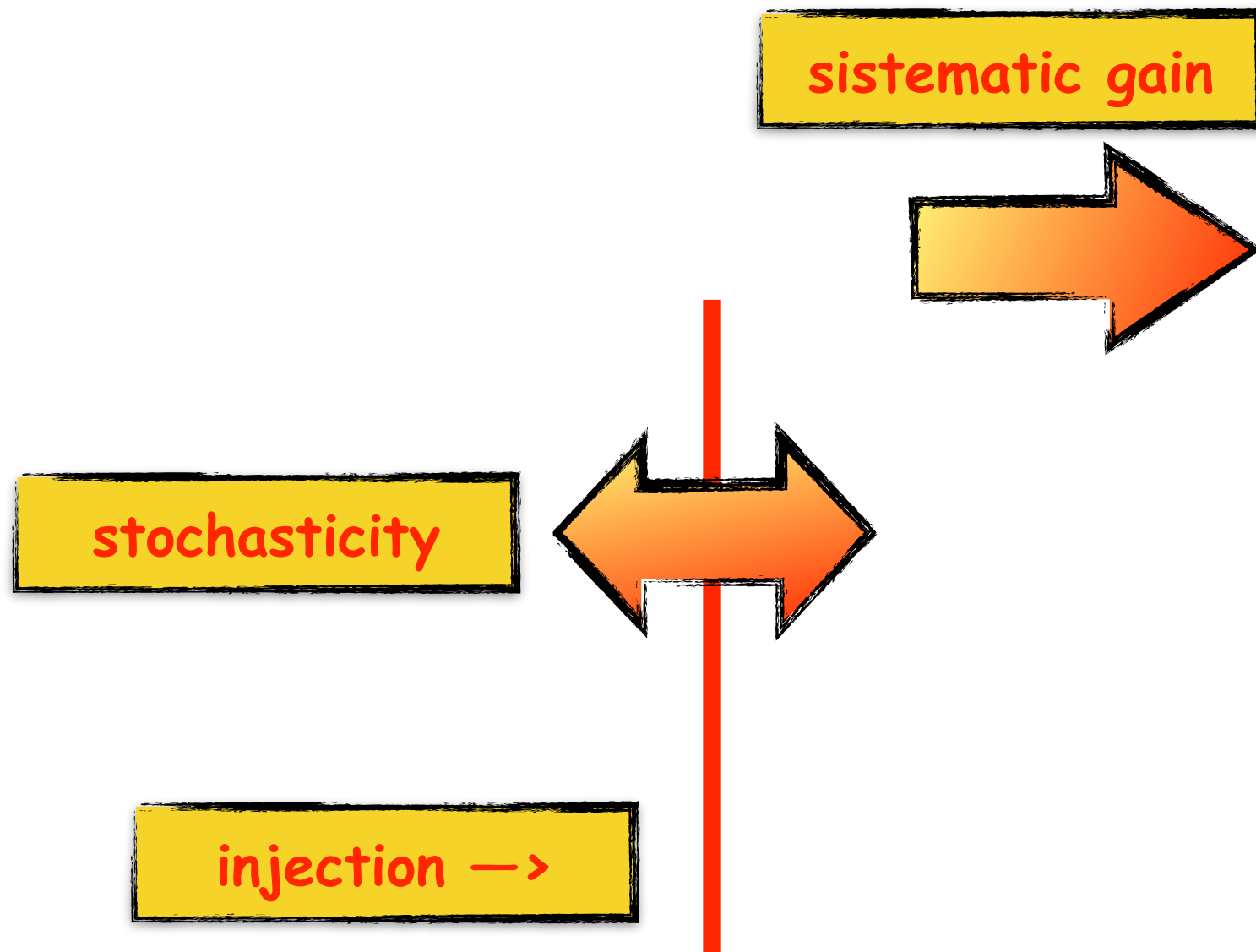
Systematic versus root mean square change in energy



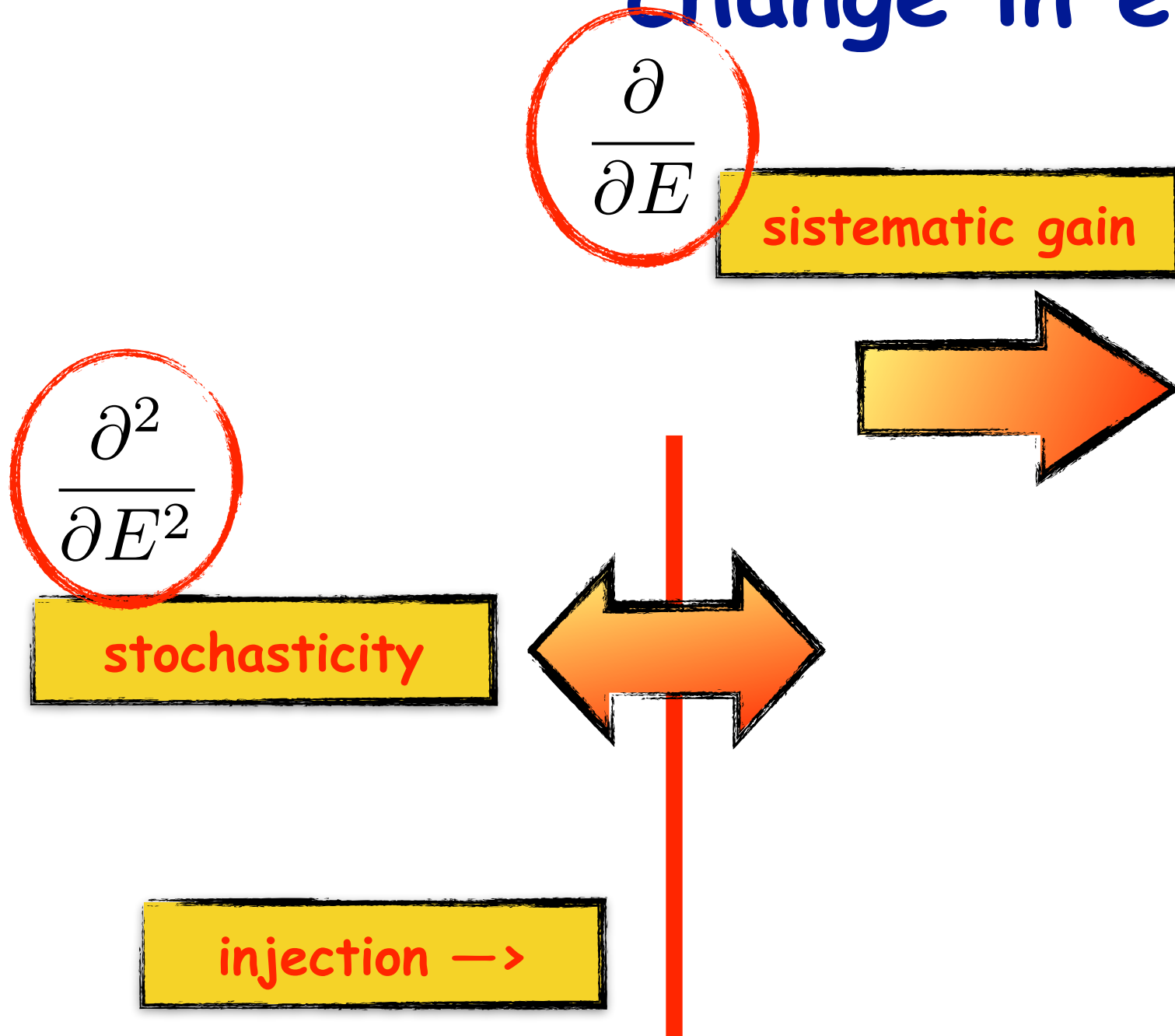
Systematic versus root mean square change in energy



Systematic versus root mean square change in energy



Systematic versus root mean square change in energy



Diffusion coefficient in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

Diffusion coefficient in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

“advection” →

$$\langle \Delta E \rangle = b(E) \Delta t$$

Diffusion coefficient in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

“advection” →

$$\langle \Delta E \rangle = b(E) \Delta t$$

diffusion coefficient

diffusion →

$$\langle (\Delta E)^2 \rangle = D_E \Delta t$$

Diffusion coefficient in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

“advection” →

$$\langle \Delta E \rangle = b(E) \Delta t$$

diffusion →

$$\langle (\Delta E)^2 \rangle = D_E \Delta t$$

diffusion coefficient

$$\tau_c = \lambda / c$$

Diffusion coefficient in energy

systematic →

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic →

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

“advection” →

$$\langle \Delta E \rangle = b(E) \Delta t$$

diffusion coefficient

$$\langle (\Delta E)^2 \rangle = D_E \Delta t$$

$$\tau_c = \lambda / c$$

diffusion →

$$D_E = \frac{\langle (\Delta E)^2 \rangle}{\Delta t} = \frac{\langle (\Delta E)^2 \rangle}{\langle \Delta E \rangle} b(e) = \frac{b(E) E}{2}$$

A more complete transport equation

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}} + \frac{1}{2} \frac{\partial^2}{\partial E^2} [D_E(E)N(E)]$$

A more complete transport equation

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}} + \frac{1}{2} \frac{\partial^2}{\partial E^2} [D_E(E)N(E)]$$

$$b(E) = E/\tau_{acc} \qquad D_E = \frac{b(E)E}{2}$$

stochasticity

A more complete transport equation

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}} + \frac{1}{2} \frac{\partial^2}{\partial E^2} [D_E(E)N(E)]$$

$$b(E) = E/\tau_{acc} \qquad D_E = \frac{b(E)E}{2}$$

stochasticity

A more complete transport equation

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}} + \frac{1}{2} \frac{\partial^2}{\partial E^2} [D_E(E)N(E)]$$

$$b(E) = E/\tau_{acc} \quad D_E = \frac{b(E)E}{2}$$

stochasticity

can we still find a power law solution? $N(E) = N_0 E^{-s}$

A more complete transport equation

$$\frac{dN(E)}{dt} = -\frac{d}{dE} [b(E)N(E)] - \frac{N(E)}{\tau_{esc}} + \frac{1}{2} \frac{\partial^2}{\partial E^2} [D_E(E)N(E)]$$

$$b(E) = E/\tau_{acc} \quad D_E = \frac{b(E)E}{2}$$

stochasticity

can we still find a power law solution? $N(E) = N_0 E^{-s}$

$$s = \frac{3}{2} \left(1 + \frac{16}{9} \frac{\tau_{acc}}{\tau_{esc}} \right)^{1/2} - \frac{1}{2}$$

Things to remember

Good things about the Fermi II mechanism

- Particles are accelerated!
- Systematic(gain) plus stochastic variation of particle energies
- Power law spectra can be generated!

Things to remember

Good things about the Fermi II mechanism

- Particles are accelerated!
- Systematic(gain) plus stochastic variation of particle energies
- Power law spectra can be generated!

Bad things about the Fermi II mechanism

- It is too slow! (second order...)
- Injection problem (in fact, this is a problem of virtually any acceleration mechanism)
- Need to be fine tuned. The slope of the power law depends on physical parameters which are a priori unknown

Fermi I, or, Diffusive Shock Acceleration

What's next

Why is the second order Fermi mechanism so slow?

What's next

Why is the second order Fermi mechanism so slow?

Maths: because it is second order! $\langle \frac{\Delta E}{E} \rangle \propto \beta^2$

What's next

Why is the second order Fermi mechanism so slow?

Maths: because it is second order! $\langle \frac{\Delta E}{E} \rangle \propto \beta^2$

What's next

Why is the second order Fermi mechanism so slow?

Maths: because it is second order! $\langle \frac{\Delta E}{E} \rangle \propto \beta^2$

Physics: because particles gain energy in some collisions, and lose in others

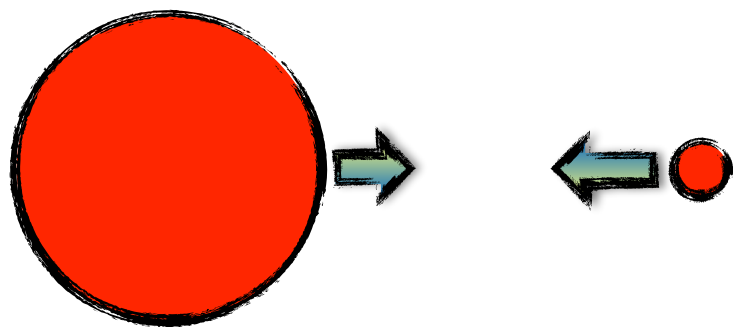
What's next

Why is the second order Fermi mechanism so slow?

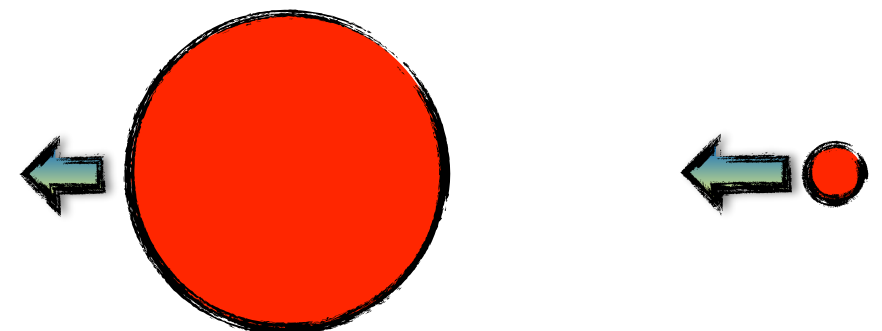
Maths: because it is second order! $\langle \frac{\Delta E}{E} \rangle \propto \beta^2$

Physics: because particles gain energy in some collisions, and lose in others

head-on collision -> gain



running after collision -> loss



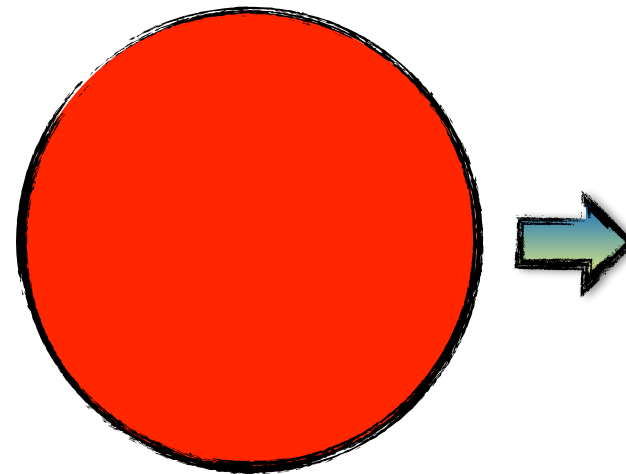
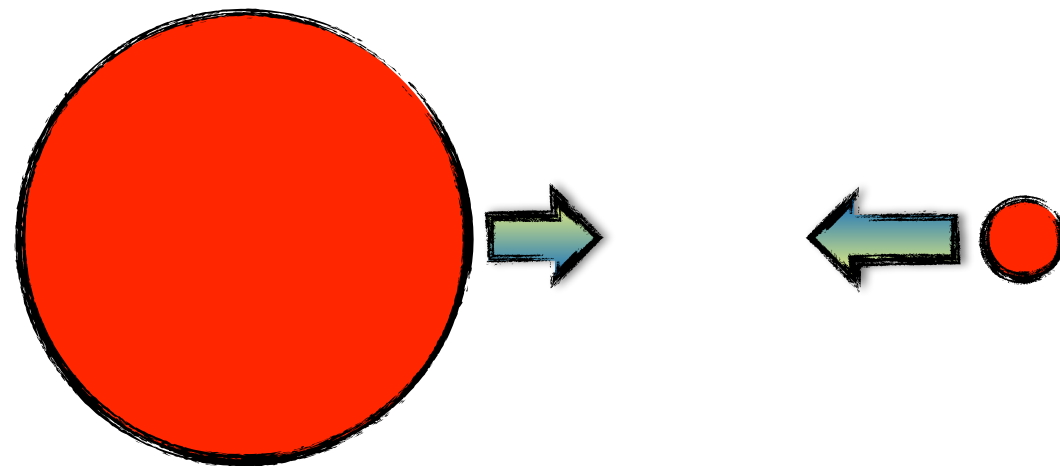
What's next

Can we do something about it?

What's next

Can we do something about it?

head-on



running
after

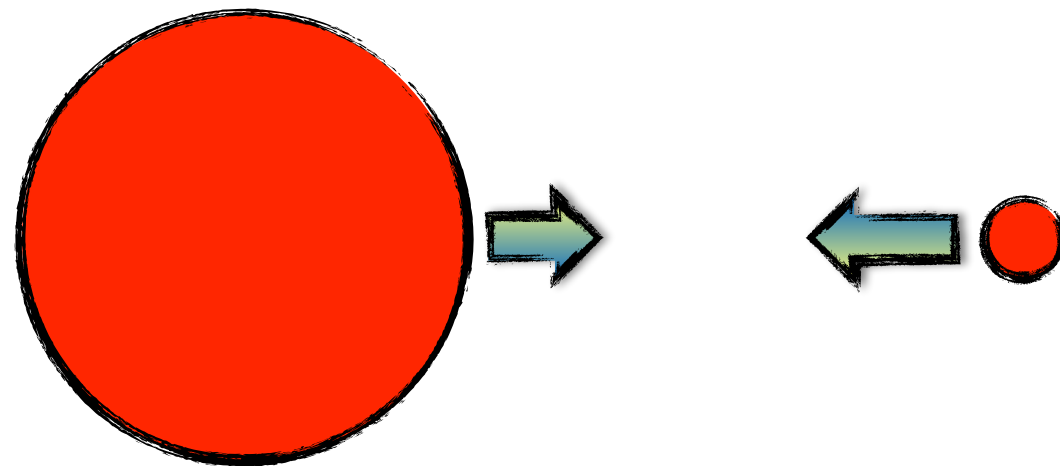
What's next

Can we do something about it?

What's next

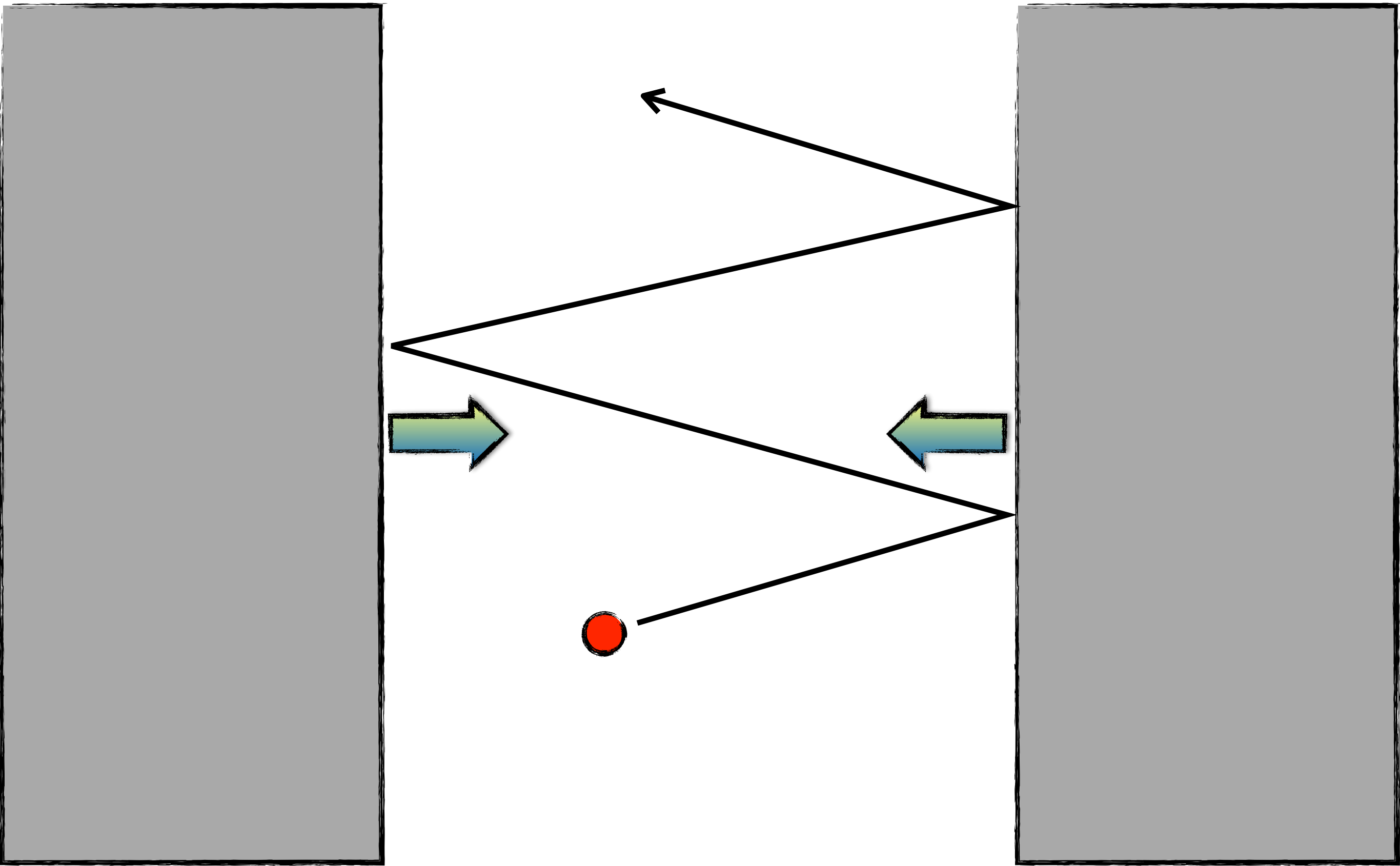
Can we do something about it?

head-on

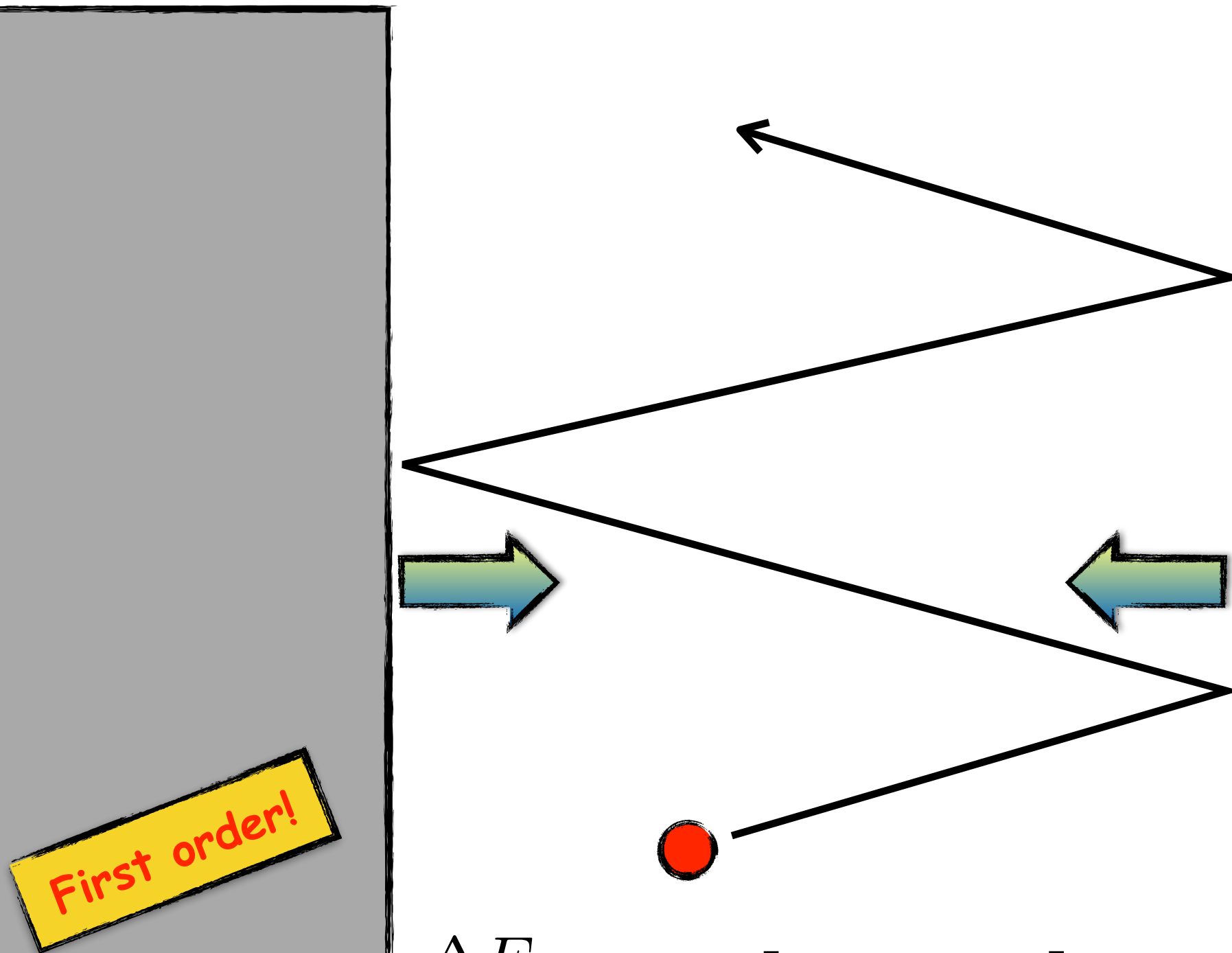


head-on

Converging walls



Converging walls



$$\left\langle \frac{\Delta E}{E} \right\rangle = 2 \frac{v}{c} \left[\cos \vartheta + \frac{v}{c} \right] \propto \frac{v}{c}$$

Shock waves

Small perturbations in a fluid: sound waves

fixed fluid element $\longrightarrow \frac{d}{dt} \longrightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \longleftarrow$ fixed position

Small perturbations in a fluid: sound waves

fixed fluid element $\longrightarrow \frac{d}{dt} \longrightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \longleftarrow$ fixed position

mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

momentum conservation

$$\rho \frac{d\vec{v}}{dt} = -\nabla P \longrightarrow \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\frac{\nabla P}{\rho}$$

Small perturbations in a fluid: sound waves

fixed fluid element $\longrightarrow \frac{d}{dt} \longrightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \longleftarrow$ fixed position

mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

momentum conservation

$$\rho \frac{d\vec{v}}{dt} = -\nabla P \longrightarrow \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\frac{\nabla P}{\rho}$$

equilibrium

$$\vec{v} = 0$$

$$\rho = \rho_0$$

$$P = P_0$$

Small perturbations in a fluid: sound waves

fixed fluid element $\longrightarrow \frac{d}{dt} \longrightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \longleftarrow$ fixed position

mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

momentum conservation

$$\rho \frac{d\vec{v}}{dt} = -\nabla P \longrightarrow \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\frac{\nabla P}{\rho}$$

equilibrium

perturbed

$$\vec{v} = 0 \longrightarrow \delta \vec{v}$$

$$\rho = \rho_0 \longrightarrow \rho_0 + \delta \rho$$

$$P = P_0 \longrightarrow P_0 + \delta P$$

Small perturbations in a fluid: sound waves

fixed fluid element $\longrightarrow \frac{d}{dt} \longrightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \longleftarrow$ fixed position

mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

momentum conservation

$$\rho \frac{d\vec{v}}{dt} = -\nabla P \longrightarrow \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\frac{\nabla P}{\rho}$$

equilibrium

perturbed

$$\vec{v} = 0 \longrightarrow \delta \vec{v}$$

$$\rho = \rho_0 \longrightarrow \rho_0 + \delta \rho$$

$$P = P_0 \longrightarrow P_0 + \delta P$$

$$\frac{\partial}{\partial t} \delta \rho + \rho_0 \nabla \cdot \delta \vec{v} = -\nabla \cdot (\delta \rho \delta \vec{v})$$

$$\frac{\partial}{\partial t} \delta \vec{v} = -\delta \vec{v} \cdot \nabla \delta \vec{v} - \frac{1}{\rho_0 + \delta \rho} \nabla \delta P$$

Small perturbations in a fluid: sound waves

fixed fluid element $\longrightarrow \frac{d}{dt} \longrightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \longleftarrow$ fixed position

mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

momentum conservation

$$\rho \frac{d\vec{v}}{dt} = -\nabla P \longrightarrow \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\frac{\nabla P}{\rho}$$

equilibrium

perturbed

$$\begin{array}{lll} \vec{v} & = & 0 \longrightarrow \delta \vec{v} \\ \rho & = & \rho_0 \longrightarrow \rho_0 + \delta \rho \\ P & = & P_0 \longrightarrow P_0 + \delta P \end{array}$$

small perturbations

$$\begin{array}{l} \frac{\partial}{\partial t} \delta \rho + \rho_0 \nabla \cdot \delta \vec{v} = -\nabla \cdot (\cancel{\rho \delta \vec{v}}) \\ \frac{\partial}{\partial t} \delta \vec{v} = -\delta \vec{v} \cdot \cancel{\nabla} \delta \vec{v} - \frac{1}{\rho_0 + \cancel{\delta \rho}} \nabla \delta P \end{array}$$

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \varrho + \varrho_0 \nabla \cdot \delta \vec{v} = 0$$

$$\varrho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\nabla \delta P$$

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \varrho + \varrho_0 \nabla \cdot \vec{\delta v} = 0 \quad \rightarrow \frac{\partial}{\partial t} \rightarrow \quad \frac{\partial^2}{\partial t^2} \delta \varrho + \varrho_0 \frac{\partial}{\partial t} \nabla \cdot \vec{\delta v} = 0$$

$$\varrho_0 \frac{\partial}{\partial t} \vec{\delta v} = -\nabla \delta P \quad \rightarrow \nabla \cdot \rightarrow \quad \varrho_0 \nabla \cdot \frac{\partial}{\partial t} \vec{\delta v} = -\nabla^2 \delta P$$

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \varrho + \varrho_0 \nabla \cdot \vec{\delta v} = 0 \quad \rightarrow \frac{\partial}{\partial t} \rightarrow \quad \frac{\partial^2}{\partial t^2} \delta \varrho + \varrho_0 \frac{\partial}{\partial t} \nabla \cdot \vec{\delta v} = 0$$

$$\varrho_0 \frac{\partial}{\partial t} \vec{\delta v} = -\nabla \delta P \quad \rightarrow \nabla \cdot \rightarrow \quad \varrho_0 \nabla \cdot \frac{\partial}{\partial t} \vec{\delta v} = -\nabla^2 \delta P$$

— subtract —>

$$\frac{\partial^2}{\partial t^2} \delta \varrho - \nabla^2 \delta P = 0$$

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \varrho + \varrho_0 \nabla \cdot \delta \vec{v} = 0 \quad \rightarrow \frac{\partial}{\partial t} \rightarrow \quad \frac{\partial^2}{\partial t^2} \delta \varrho + \varrho_0 \frac{\partial}{\partial t} \nabla \cdot \delta \vec{v} = 0$$

$$\varrho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\nabla \delta P \quad \rightarrow \nabla \cdot \rightarrow \quad \varrho_0 \nabla \cdot \frac{\partial}{\partial t} \delta \vec{v} = -\nabla^2 \delta P$$

— subtract —>

adiabatic eq. of state

$$P = K \varrho^\gamma$$

$$\gamma = \frac{5}{3} \quad \text{monoatomic gas}$$

$$\delta P = \left(\frac{\partial P}{\partial \varrho} \right)_s \delta \varrho = \frac{\gamma P}{\varrho} \delta \varrho \equiv c_s^2 \delta \varrho$$

constant entropy

sound speed

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \varrho + \varrho_0 \nabla \cdot \delta \vec{v} = 0 \quad \rightarrow \frac{\partial}{\partial t} \rightarrow \quad \frac{\partial^2}{\partial t^2} \delta \varrho + \varrho_0 \frac{\partial}{\partial t} \nabla \cdot \delta \vec{v} = 0$$

$$\varrho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\nabla \delta P \quad \rightarrow \nabla \cdot \rightarrow \quad \varrho_0 \nabla \cdot \frac{\partial}{\partial t} \delta \vec{v} = -\nabla^2 \delta P$$

— subtract —→

adiabatic eq. of state

$$P = K \varrho^\gamma$$

$$\gamma = \frac{5}{3} \quad \text{monoatomic gas}$$

$$\delta P = \left(\frac{\partial P}{\partial \varrho} \right)_s \delta \varrho = \frac{\gamma P}{\varrho} \delta \varrho \equiv c_s^2 \delta \varrho$$

constant entropy

sound speed

small perturbations

$$c_{s,0}^2 = \frac{\gamma P_0}{\varrho_0}$$

$$\frac{\partial^2}{\partial t^2} \delta \varrho - \nabla^2 \delta P = 0$$

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \rho + \rho_0 \nabla \cdot \delta \vec{v} = 0 \quad \rightarrow \frac{\partial}{\partial t} \rightarrow \quad \frac{\partial^2}{\partial t^2} \delta \rho + \rho_0 \frac{\partial}{\partial t} \nabla \cdot \delta \vec{v} = 0$$

$$\rho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\nabla \delta P \quad \rightarrow \nabla \cdot \rightarrow \quad \rho_0 \nabla \cdot \frac{\partial}{\partial t} \delta \vec{v} = -\nabla^2 \delta P$$

— subtract —→

adiabatic eq. of state

$$P = K \rho^\gamma$$

$$\gamma = \frac{5}{3} \quad \text{monoatomic gas}$$

$$\delta P = \left(\frac{\partial P}{\partial \rho} \right)_s \delta \rho = \frac{\gamma P}{\rho} \delta \rho \equiv c_s^2 \delta \rho$$

constant entropy

sound speed

small perturbations

$$c_{s,0}^2 = \frac{\gamma P_0}{\rho_0}$$

$$\frac{\partial^2}{\partial t^2} \delta \rho - \nabla^2 \delta P = 0$$

wave equation

$$\frac{\partial^2}{\partial t^2} \delta \rho - c_{s,0}^2 \nabla^2 \delta \rho = 0$$

Small perturbations in a fluid: sound waves

$$\frac{\partial}{\partial t} \delta \rho + \rho_0 \nabla \cdot \delta \vec{v} = 0 \quad \rightarrow \frac{\partial}{\partial t} \rightarrow \quad \frac{\partial^2}{\partial t^2} \delta \rho + \rho_0 \frac{\partial}{\partial t} \nabla \cdot \delta \vec{v} = 0$$

$$\rho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\nabla \delta P \quad \rightarrow \nabla \cdot \rightarrow \quad \rho_0 \nabla \cdot \frac{\partial}{\partial t} \delta \vec{v} = -\nabla^2 \delta P$$

— subtract —→

adiabatic eq. of state

$$P = K \rho^\gamma$$

$$\gamma = \frac{5}{3} \quad \text{monoatomic gas}$$

$$\delta P = \left(\frac{\partial P}{\partial \rho} \right)_s \delta \rho = \frac{\gamma P}{\rho} \delta \rho \equiv c_s^2 \delta \rho$$

constant entropy

sound speed

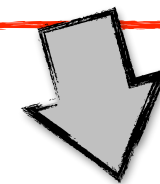
$$c_{s,0}^2 = \frac{\gamma P_0}{\rho_0}$$

small perturbations

$$\frac{\partial^2}{\partial t^2} \delta \rho - \nabla^2 \delta P = 0$$

wave equation

$$\frac{\partial^2}{\partial t^2} \delta \rho - c_{s,0}^2 \nabla^2 \delta \rho = 0$$



$$\frac{\delta \rho}{\rho_0} = A e^{i(\vec{k} \cdot \vec{x} \pm \omega t)}$$

$$A \ll 1, \quad \omega^2 = k^2 c_{s,0}^2$$

The formation of shock waves

let's relax the assumption of small perturbations $\rightarrow$ finite amplitude perturbations

$$c_s \sim c_{s,0}$$

The formation of shock waves

let's relax the assumption of small perturbations → finite amplitude perturbations

$$c_s \not> c_{s,0}$$

supersonic

$$c_s > c_{s,0}$$

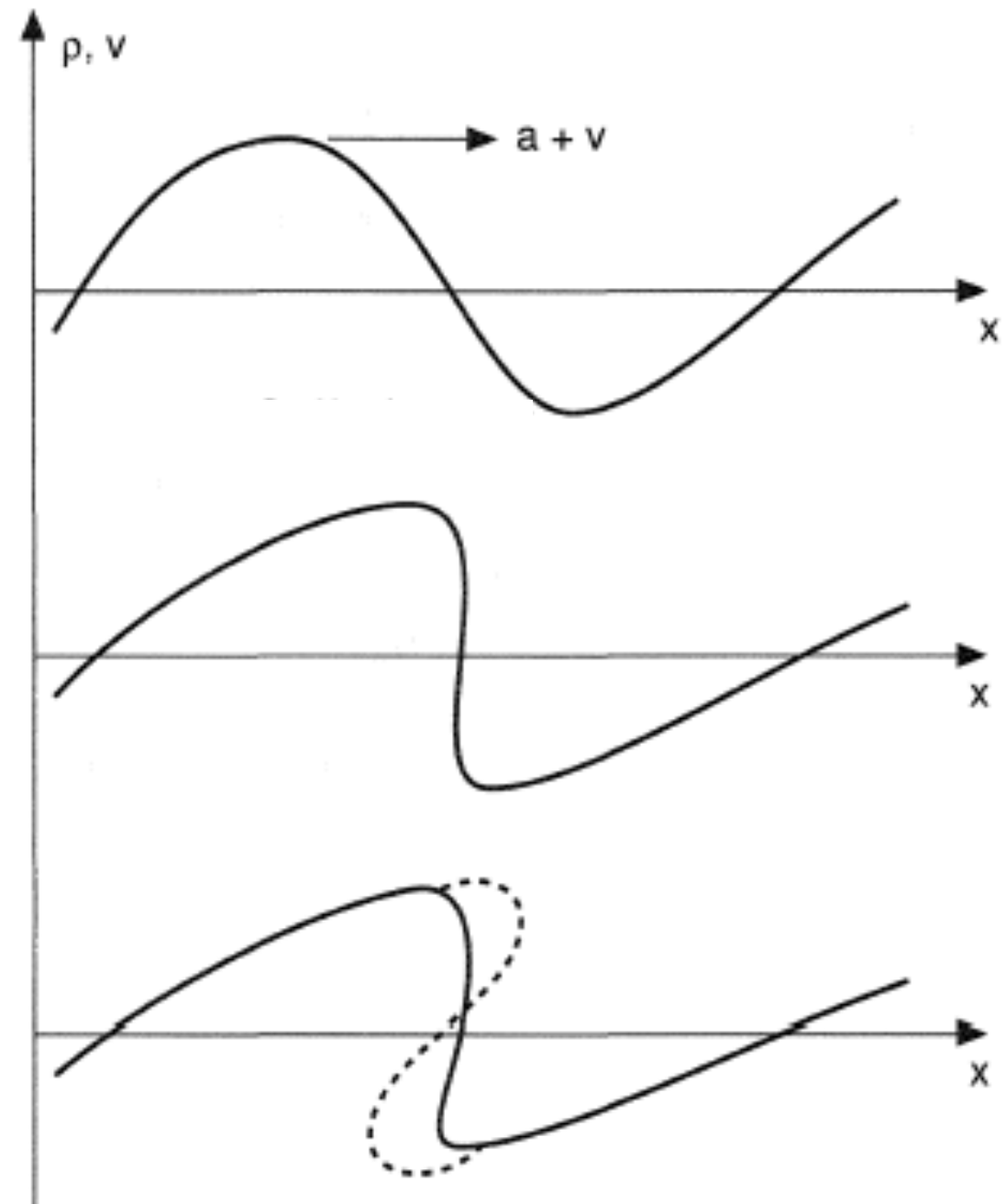
The formation of shock waves

let's relax the assumption of small perturbations $\rightarrow$ finite amplitude perturbations

$$c_s \not> c_{s,0}$$

supersonic

$$c_s > c_{s,0}$$



The formation of shock waves

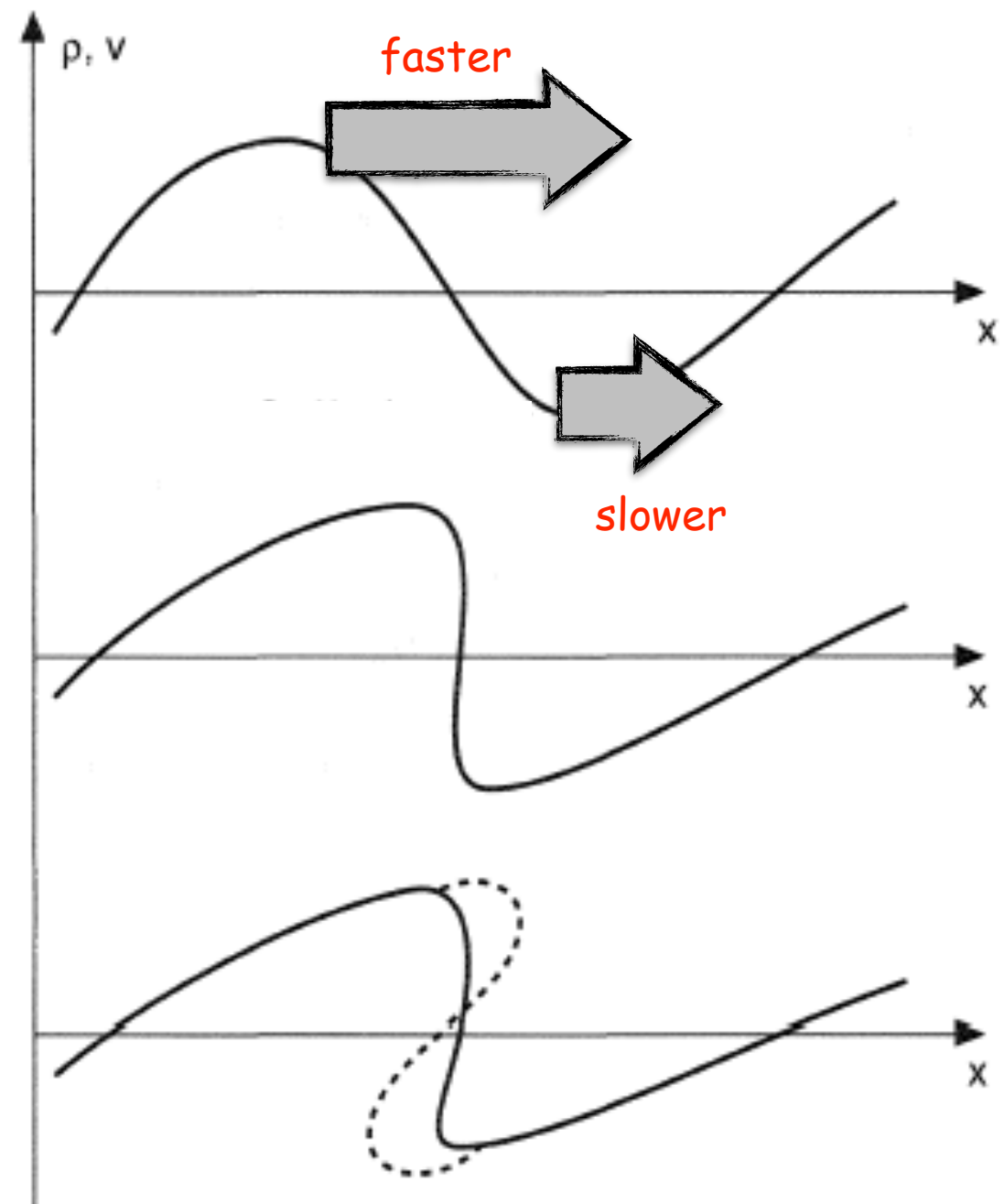
let's relax the assumption of small perturbations → finite amplitude perturbations

$$c_s \not> c_{s,0}$$

supersonic

$$c_s > c_{s,0}$$

the propagation speed is faster when the density is larger



The formation of shock waves

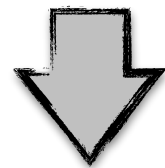
let's relax the assumption of small perturbations → finite amplitude perturbations

$$c_s \not> c_{s,0}$$

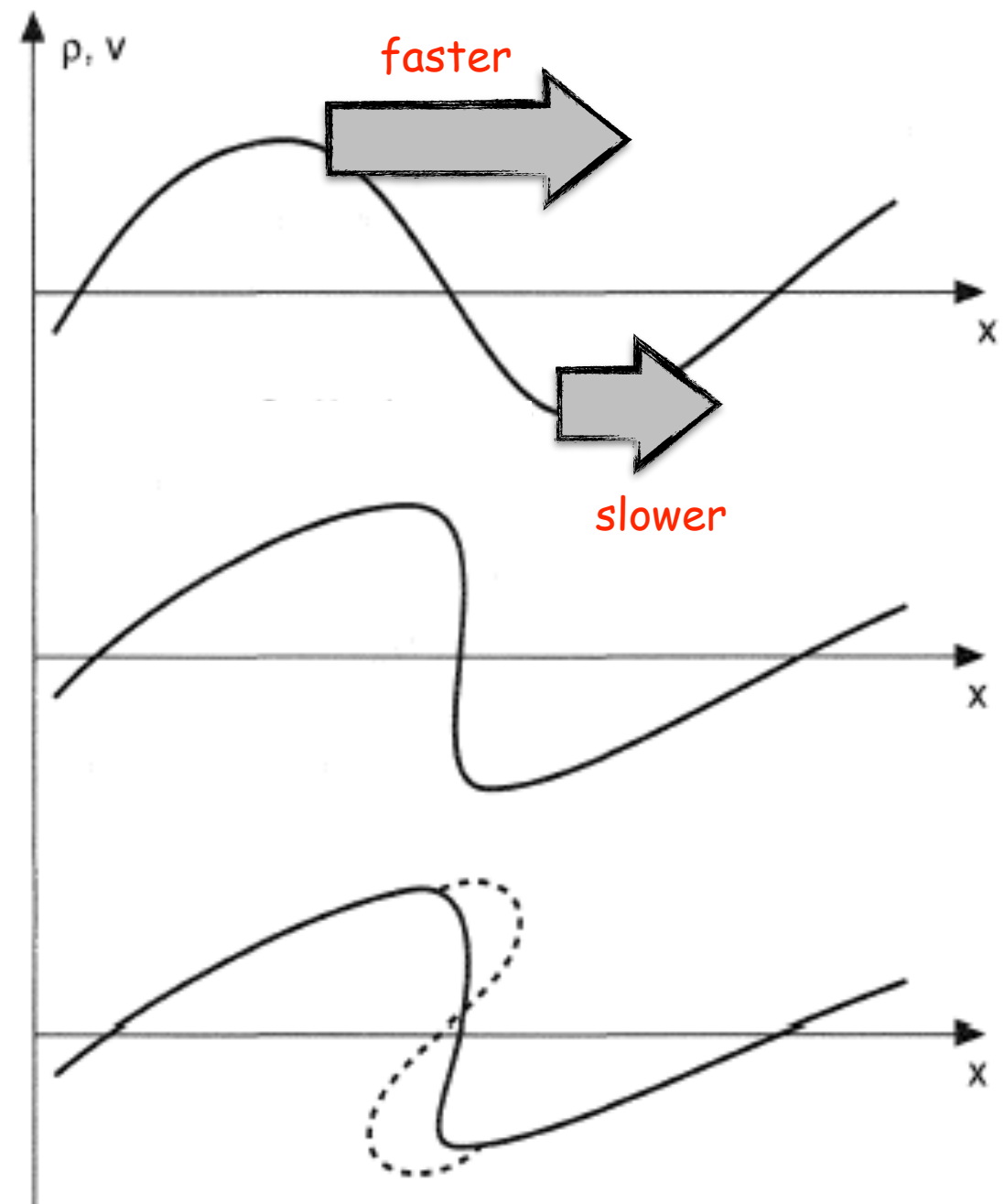
supersonic

$$c_s > c_{s,0}$$

the propagation speed is faster when the density is larger



steepening of the wave

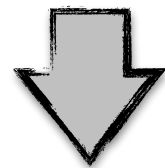


The formation of shock waves

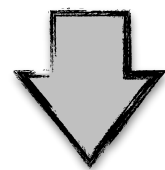
let's relax the assumption of small perturbations → finite amplitude perturbations

~~c_s~~ $c_{s,0}$ ^{supersonic} $c_s > c_{s,0}$

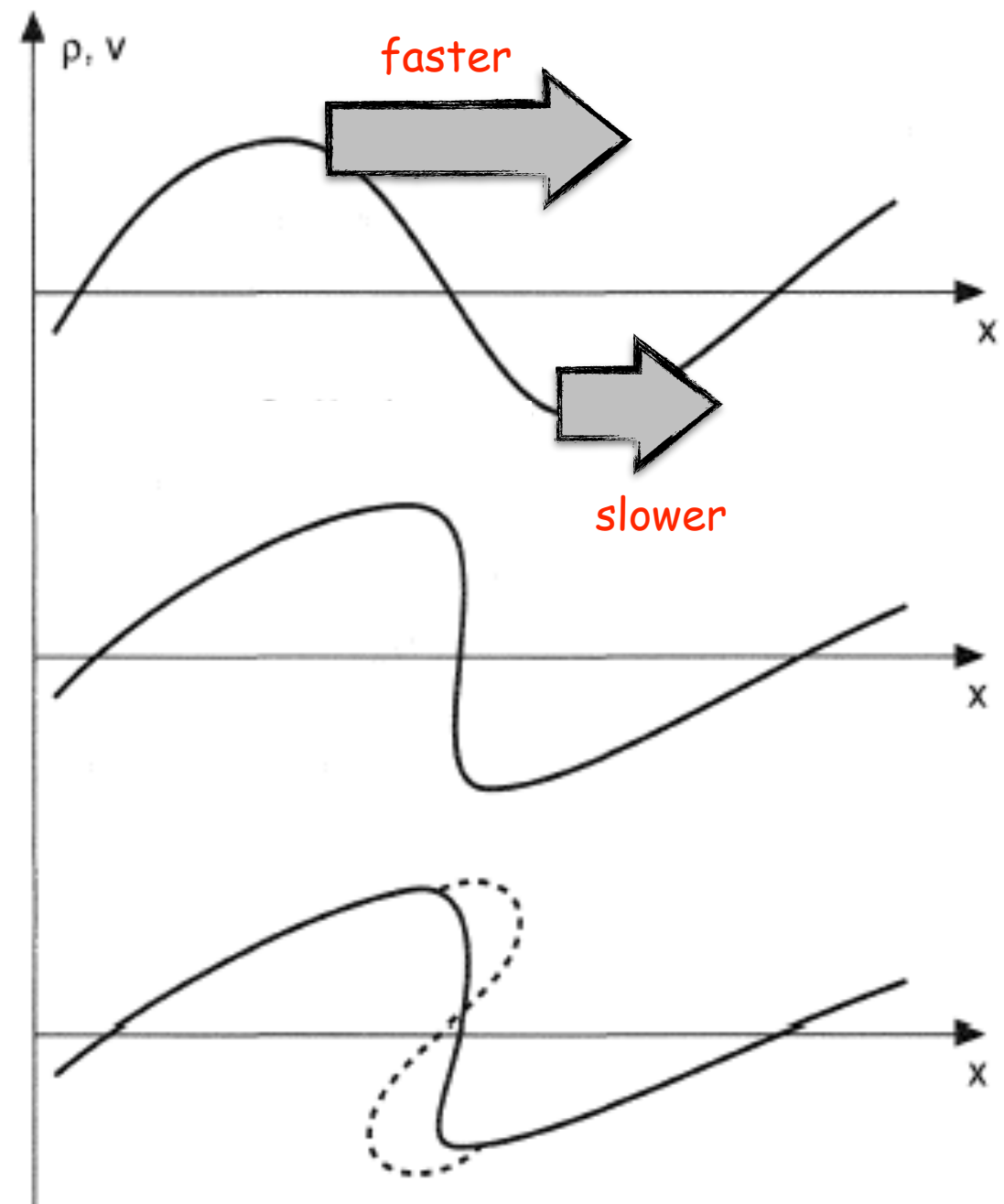
the propagation speed is faster when the density is larger



steepening of the wave

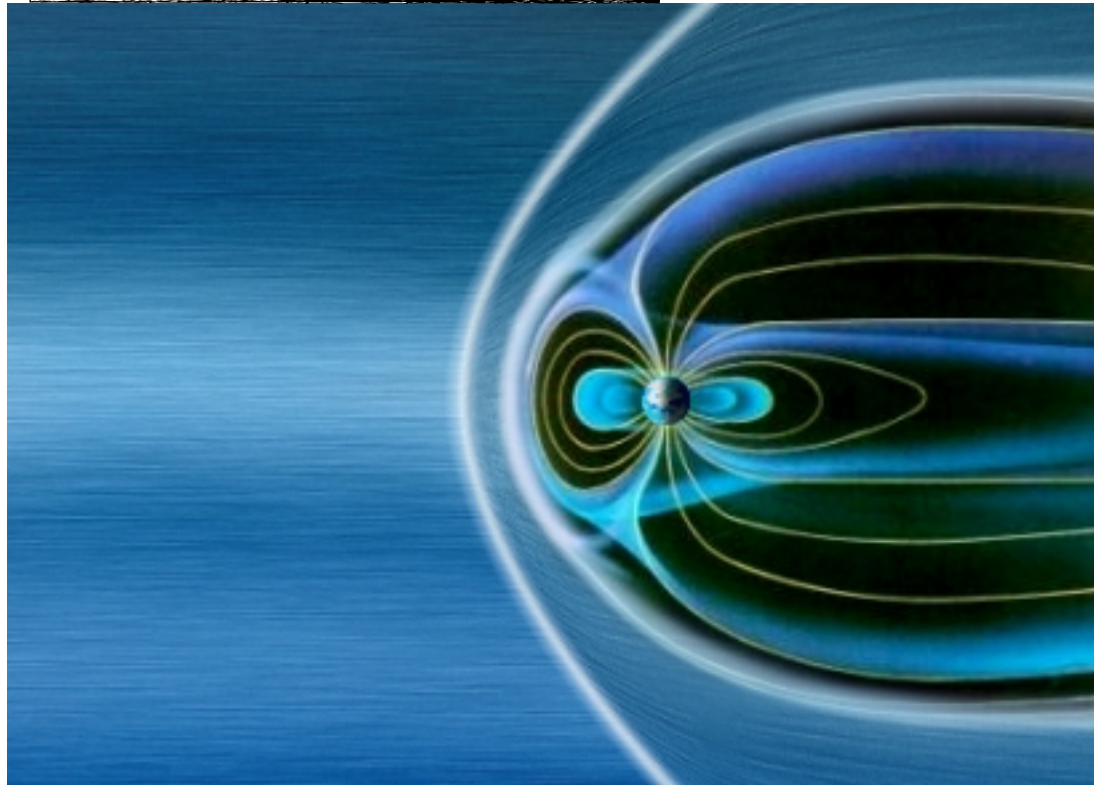


formation of a discontinuity

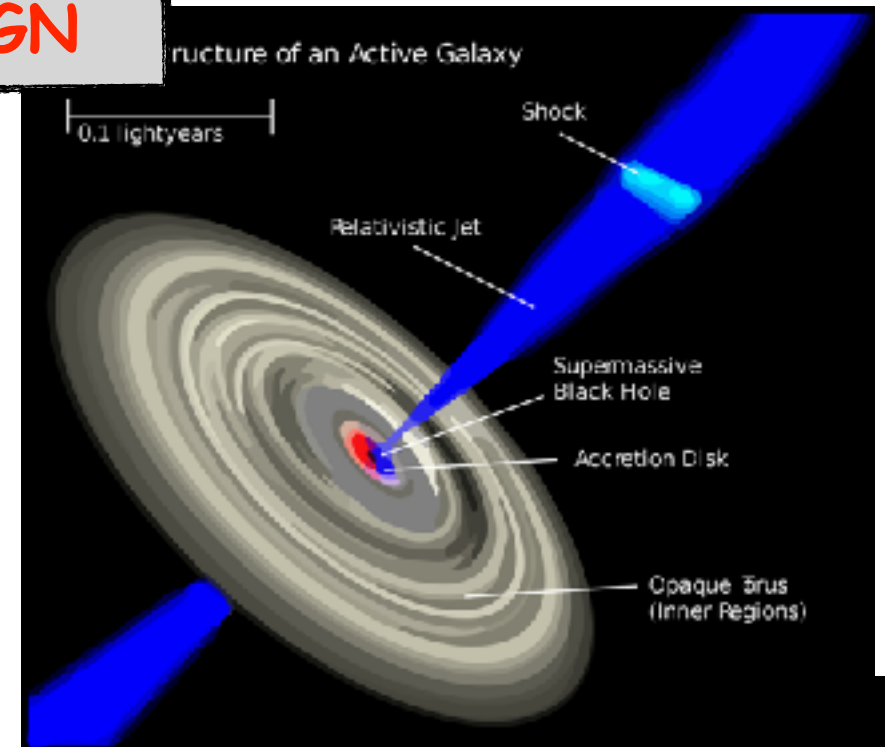


Shocks in the Universe

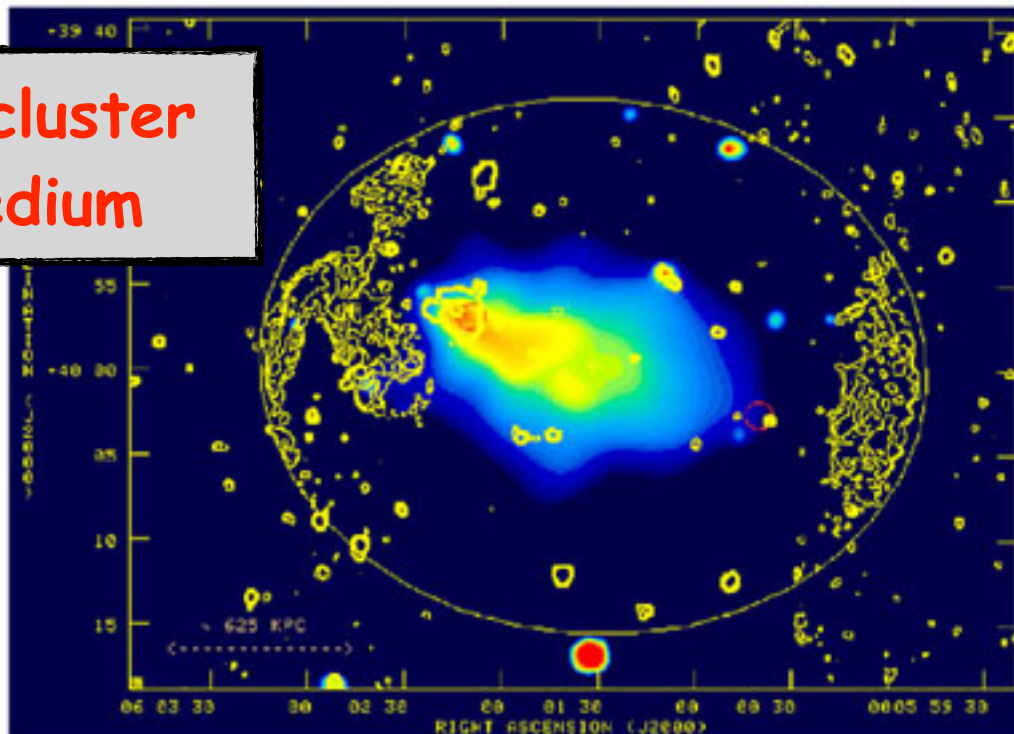
Earth bow shock



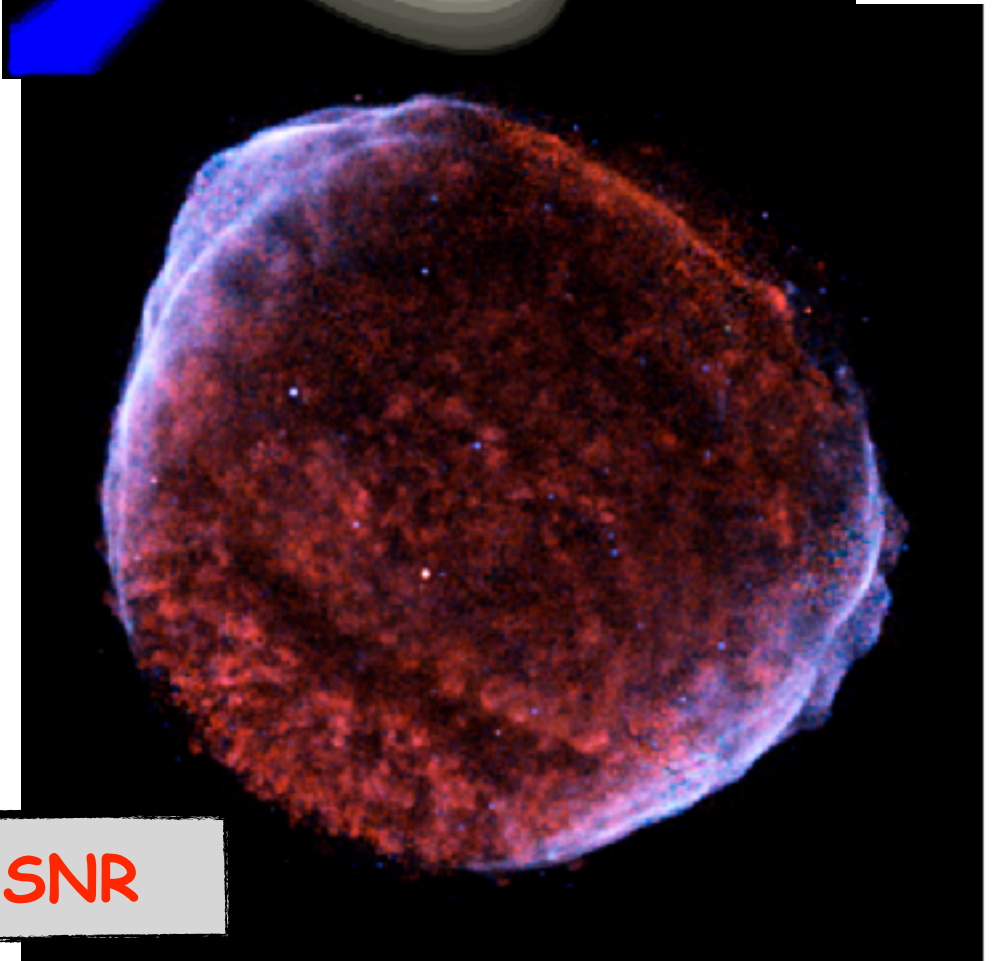
AGN



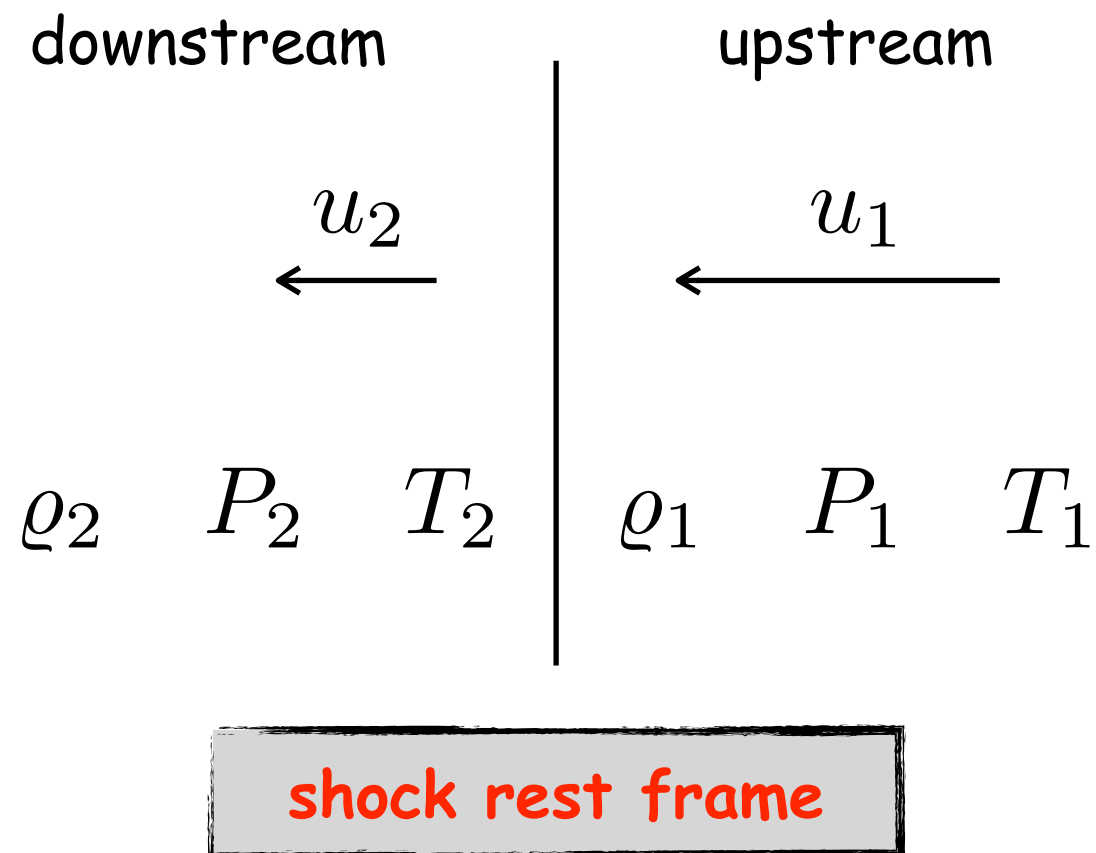
intracluster medium



SNR



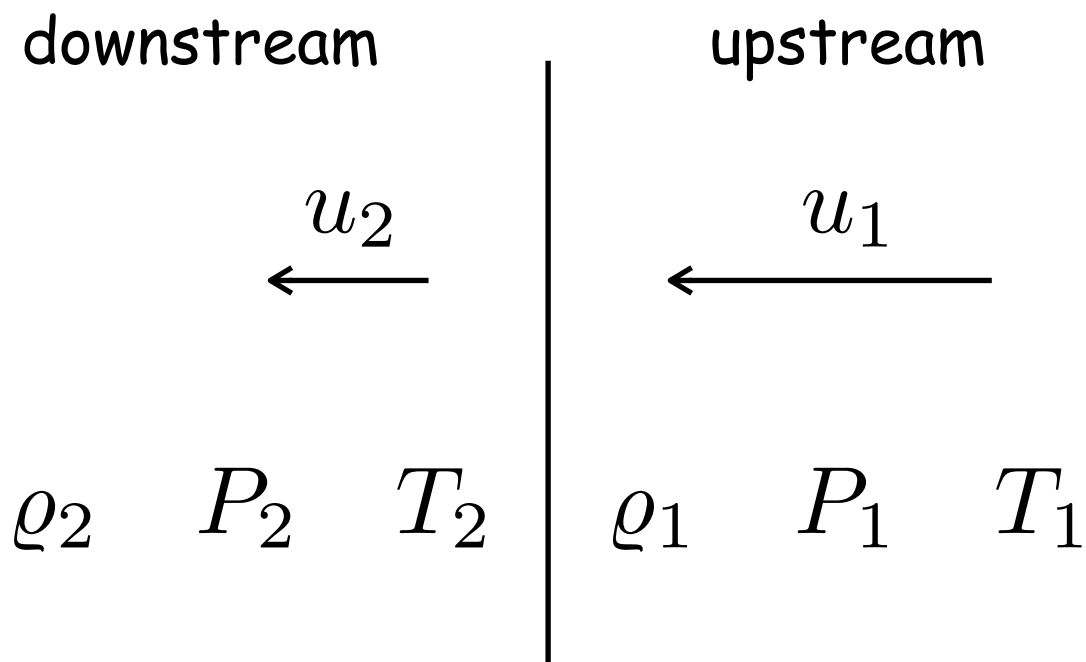
Shock waves: conservation laws



Mach number

$$\mathcal{M} = \frac{u_1}{c_s}$$

Shock waves: conservation laws



Mach number

$$\mathcal{M} = \frac{u_1}{c_s}$$

shock rest frame

mass

$$\rho_1 u_1 = \rho_2 u_2$$

momentum

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

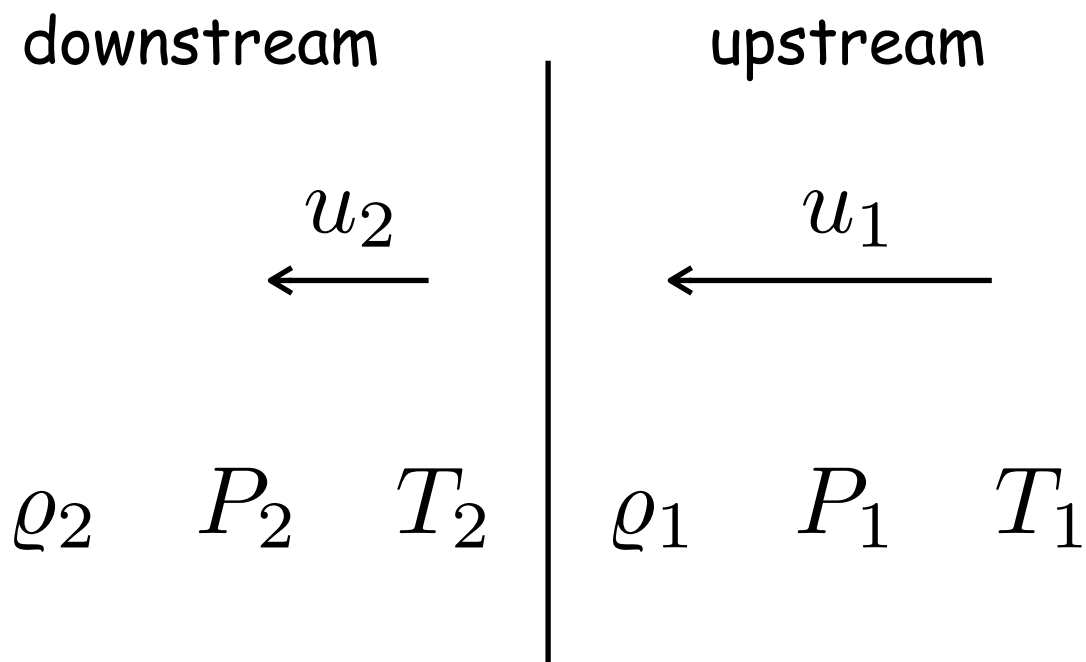
energy

$$\frac{1}{2} \rho_1 u_1^3 + \rho_1 u_1 \left(\epsilon_1 + \frac{P_1}{\rho_1} \right) = \frac{1}{2} \rho_2 u_2^3 + \rho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\rho_2} \right)$$

internal energy per unit mass

specific enthalpy

Shock waves: conservation laws



Mach number

$$\mathcal{M} = \frac{u_1}{c_s}$$

shock rest frame

mass

$$\rho_1 u_1 = \rho_2 u_2$$

momentum

$$\left. \begin{aligned} \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 \end{aligned} \right\} 1 + \frac{P_1}{\rho_1 u_1^2} = \frac{u_2}{u_1} + \frac{P_2}{\rho_1 u_1^2}$$

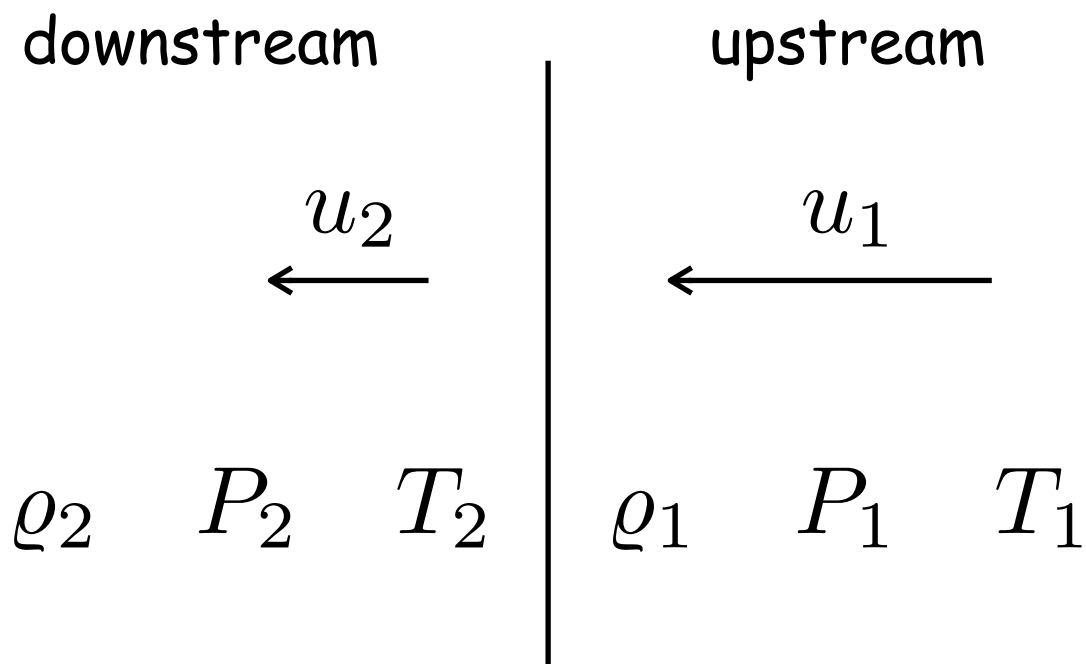
energy

$$\frac{1}{2} \rho_1 u_1^3 + \rho_1 u_1 \left(\epsilon_1 + \frac{P_1}{\rho_1} \right) = \frac{1}{2} \rho_2 u_2^3 + \rho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\rho_2} \right)$$

internal energy per unit mass

specific enthalpy

Shock waves: conservation laws



shock rest frame

Mach number

$$\mathcal{M} = \frac{u_1}{c_s}$$

strong shock

$$\mathcal{M}^2 \gg 1 \longrightarrow \frac{\rho_1 u_1^2}{\gamma P_1} \gg 1$$

mass

$$\rho_1 u_1 = \rho_2 u_2$$

momentum

$$\left. \begin{aligned} \rho_1 u_1^2 + P_1 &= \rho_2 u_2^2 + P_2 \end{aligned} \right\} 1 + \frac{P_1}{\rho_1 u_1^2} = \frac{u_2}{u_1} + \frac{P_2}{\rho_1 u_1^2}$$

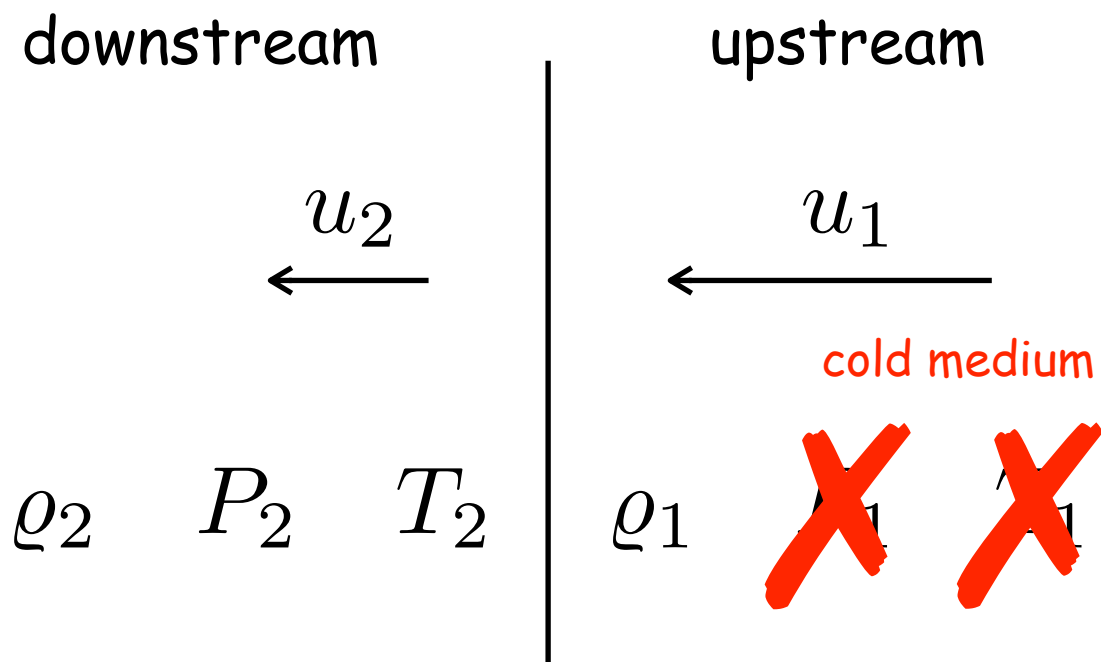
energy

$$\frac{1}{2} \rho_1 u_1^3 + \rho_1 u_1 \left(\epsilon_1 + \frac{P_1}{\rho_1} \right) = \frac{1}{2} \rho_2 u_2^3 + \rho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\rho_2} \right)$$

internal energy per unit mass

specific enthalpy

Shock waves: conservation laws



Mach number

$$\mathcal{M} = \frac{u_1}{c_s}$$

shock rest frame

strong shock

$$\mathcal{M}^2 \gg 1 \longrightarrow \frac{\rho_1 u_1^2}{\gamma P_1} \gg 1$$

mass

$$\rho_1 u_1 = \rho_2 u_2$$

momentum

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$1 + \frac{P_2}{\rho_1 u_1^2} = \frac{u_2}{u_1} + \frac{P_2}{\rho_1 u_1^2}$$

energy

$$\frac{1}{2} \rho_1 u_1^3 + \rho_1 u_1 \left(\epsilon_1 + \frac{P_1}{\rho_1} \right) = \frac{1}{2} \rho_2 u_2^3 + \rho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\rho_2} \right)$$

internal energy per unit mass

specific enthalpy

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

$$\begin{cases} \frac{P_2}{\varrho_1 u_1^2} = 1 - \frac{1}{r} \\ \frac{1}{2} \varrho_1 u_1^3 = \frac{1}{2} \varrho_2 u_2^3 + \varrho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\varrho_2} \right) \end{cases}$$

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

$$\begin{cases} \frac{P_2}{\varrho_1 u_1^2} = 1 - \frac{1}{r} \\ \frac{1}{2} \varrho_1 u_1^3 = \frac{1}{2} \varrho_2 u_2^3 + \varrho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\varrho_2} \right) \rightarrow \varrho_1 u_1^3 = \varrho_2 u_2^3 + \frac{2\gamma}{\gamma-1} P_2 u_2 \end{cases}$$

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

$$\begin{cases} \frac{P_2}{\varrho_1 u_1^2} = 1 - \frac{1}{r} \\ \frac{1}{2} \varrho_1 u_1^3 = \frac{1}{2} \varrho_2 u_2^3 + \varrho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\varrho_2} \right) \rightarrow \varrho_1 u_1^3 = \varrho_2 u_2^3 + \frac{2\gamma}{\gamma-1} P_2 u_2 \end{cases}$$

$$\rightarrow \frac{u_1^2}{u_2^2} = 1 + \frac{2\gamma}{\gamma-1} \left(\frac{P_2}{\varrho_1 u_1^2} \right) \left(\frac{u_1}{u_2} \right)$$

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

$$\begin{cases} \frac{P_2}{\varrho_1 u_1^2} = 1 - \frac{1}{r} \\ \frac{1}{2} \varrho_1 u_1^3 = \frac{1}{2} \varrho_2 u_2^3 + \varrho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\varrho_2} \right) \rightarrow \varrho_1 u_1^3 = \varrho_2 u_2^3 + \frac{2\gamma}{\gamma-1} P_2 u_2 \end{cases}$$

$$\rightarrow \frac{u_1^2}{u_2^2} = 1 + \frac{2\gamma}{\gamma-1} \left(\frac{P_2}{\varrho_1 u_1^2} \right) \left(\frac{u_1}{u_2} \right) \rightarrow r^2 - 1 = \frac{2\gamma}{\gamma-1} (r - 1)$$

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

$$\begin{cases} \frac{P_2}{\varrho_1 u_1^2} = 1 - \frac{1}{r} \\ \frac{1}{2} \varrho_1 u_1^3 = \frac{1}{2} \varrho_2 u_2^3 + \varrho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\varrho_2} \right) \rightarrow \varrho_1 u_1^3 = \varrho_2 u_2^3 + \frac{2\gamma}{\gamma-1} P_2 u_2 \end{cases}$$

$$\rightarrow \frac{u_1^2}{u_2^2} = 1 + \frac{2\gamma}{\gamma-1} \left(\frac{P_2}{\varrho_1 u_1^2} \right) \left(\frac{u_1}{u_2} \right) \rightarrow r^2 - 1 = \frac{2\gamma}{\gamma-1} (r - 1)$$

 $r = 1 \rightarrow$ unphysical solution

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \varrho_2$$

$$\begin{cases} \frac{P_2}{\varrho_1 u_1^2} = 1 - \frac{1}{r} \\ \frac{1}{2} \varrho_1 u_1^3 = \frac{1}{2} \varrho_2 u_2^3 + \varrho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\varrho_2} \right) \rightarrow \varrho_1 u_1^3 = \varrho_2 u_2^3 + \frac{2\gamma}{\gamma-1} P_2 u_2 \end{cases}$$

$$\rightarrow \frac{u_1^2}{u_2^2} = 1 + \frac{2\gamma}{\gamma-1} \left(\frac{P_2}{\varrho_1 u_1^2} \right) \left(\frac{u_1}{u_2} \right) \rightarrow r^2 - 1 = \frac{2\gamma}{\gamma-1} (r - 1)$$

$r = 1 \rightarrow$ unphysical solution

$$\rightarrow r = \frac{2\gamma}{\gamma-1} - 1 = \frac{\gamma+1}{\gamma-1} \xrightarrow{\gamma=5/3} 4 \quad \text{gas compression}$$

Jump (Rankine-Hugoniot) conditions

compression factor

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$$

ideal gas

$$P_2 = (\gamma - 1)\epsilon_2 \rho_2$$

$$\begin{cases} \frac{P_2}{\rho_1 u_1^2} = 1 - \frac{1}{r} \rightarrow P_2 = \frac{2}{\gamma+1} \rho_1 u_1^2 \xrightarrow{\gamma=5/3} \frac{3}{4} \rho_1 u_1^2 \text{ ram-pressure} \rightarrow \text{thermal pressure} \\ \frac{1}{2} \rho_1 u_1^3 = \frac{1}{2} \rho_2 u_2^3 + \rho_2 u_2 \left(\epsilon_2 + \frac{P_2}{\rho_2} \right) \rightarrow \rho_1 u_1^3 = \rho_2 u_2^3 + \frac{2\gamma}{\gamma-1} P_2 u_2 \end{cases}$$

$$\rightarrow \frac{u_1^2}{u_2^2} = 1 + \frac{2\gamma}{\gamma-1} \left(\frac{P_2}{\rho_1 u_1^2} \right) \left(\frac{u_1}{u_2} \right) \rightarrow r^2 - 1 = \frac{2\gamma}{\gamma-1} (r - 1)$$

$r = 1 \rightarrow$ unphysical solution

$$\rightarrow r = \frac{2\gamma}{\gamma-1} - 1 = \frac{\gamma+1}{\gamma-1} \xrightarrow{\gamma=5/3} 4 \quad \text{gas compression}$$

Strong shocks

- compress moderately the gas $r = \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \xrightarrow{\gamma=5/3} 4$

Strong shocks

■ compress moderately the gas $r = \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \xrightarrow{\gamma=5/3} 4$

■ make the downstream gas subsonic

$$\mathcal{M}_2^2 = \frac{u_2^2}{c_{s,2}^2} = \frac{\rho_2 u_2^2}{\gamma P_2} = \frac{\gamma-1}{2\gamma} \xrightarrow{\gamma=5/3} \frac{1}{5} < 1$$

Strong shocks

- compress moderately the gas $r = \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \xrightarrow{\gamma=5/3} 4$

- make the downstream gas subsonic

$$\mathcal{M}_2^2 = \frac{u_2^2}{c_{s,2}^2} = \frac{\rho_2 u_2^2}{\gamma P_2} = \frac{\gamma-1}{2\gamma} \xrightarrow{\gamma=5/3} \frac{1}{5} < 1$$

- heat significantly the gas

$$P_2 = \frac{2}{\gamma+1} \rho_1 u_1^2 \equiv \frac{\rho_2}{m_p} k T_2$$

Rankine-Hugoniot conditions

So far we considered strong shocks only.

If we relax this assumption we get the following jump conditions:

compression factor

$$r = \frac{4\mathcal{M}^2}{\mathcal{M}^2 + 3} \leq 4$$

pressure

$$\frac{P_2}{P_1} = \frac{5\mathcal{M}^2 - 1}{4}$$

temperature

$$\frac{T_2}{T_1} = \frac{(5\mathcal{M}^2 - 1)(\mathcal{M}^2 + 3)}{16\mathcal{M}^2}$$

Rankine-Hugoniot conditions

So far we considered strong shocks only.

If we relax this assumption we get the following jump conditions:

compression factor

$$r = \frac{4\mathcal{M}^2}{\mathcal{M}^2 + 3} \leq 4$$

pressure

$$\frac{P_2}{P_1} = \frac{5\mathcal{M}^2 - 1}{4}$$

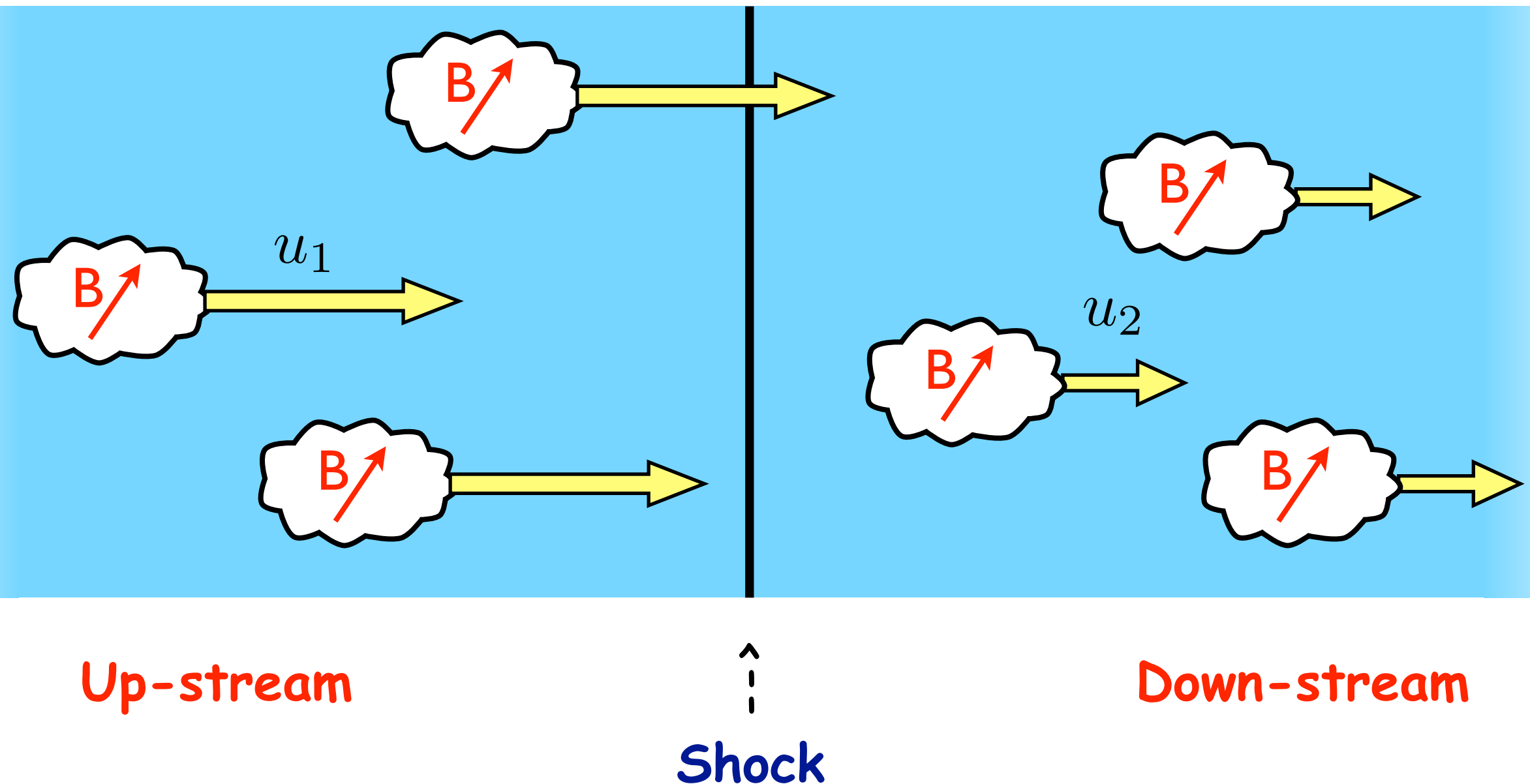
temperature

$$\frac{T_2}{T_1} = \frac{(5\mathcal{M}^2 - 1)(\mathcal{M}^2 + 3)}{16\mathcal{M}^2}$$

**Diffusive shock
acceleration**

Diffusive Shock Acceleration

Shock rest frame

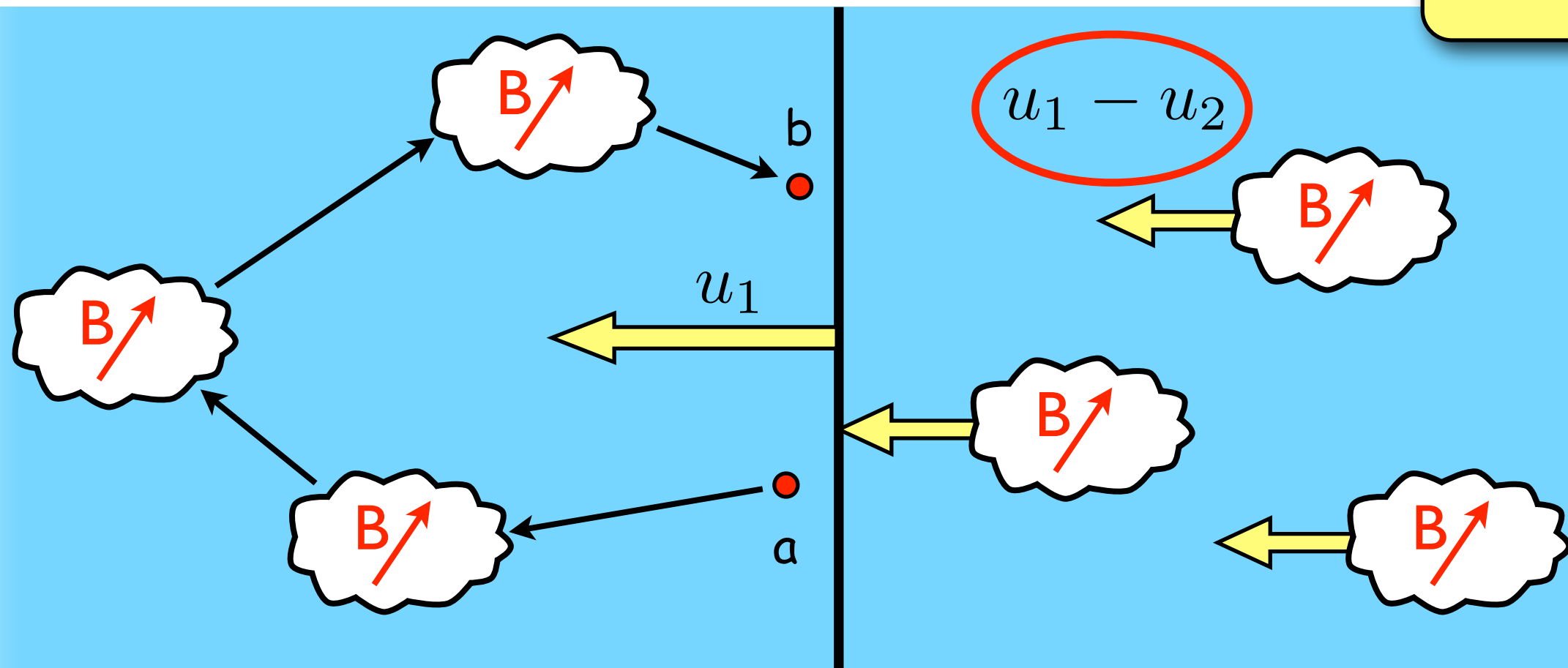


Krymskii 1977, Axford et al. 1977, Blandford & Ostriker 1978, Bell 1978

Diffusive Shock Acceleration

Up-stream rest frame

$$E_a = E_b$$



Up-stream

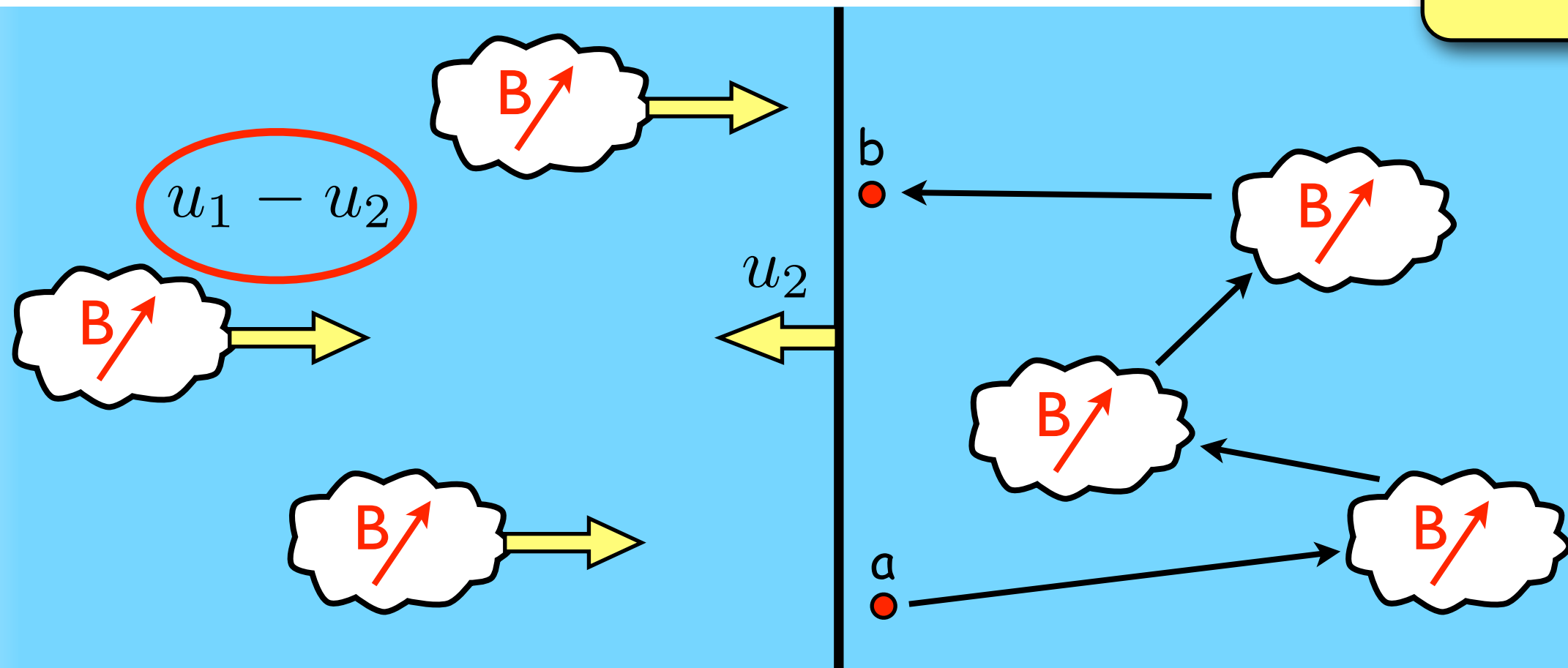
Shock

Down-stream

Diffusive Shock Acceleration

Down-stream rest frame

$$E_a = E_b$$



Up-stream

Shock

Down-stream

Diffusive Shock Acceleration

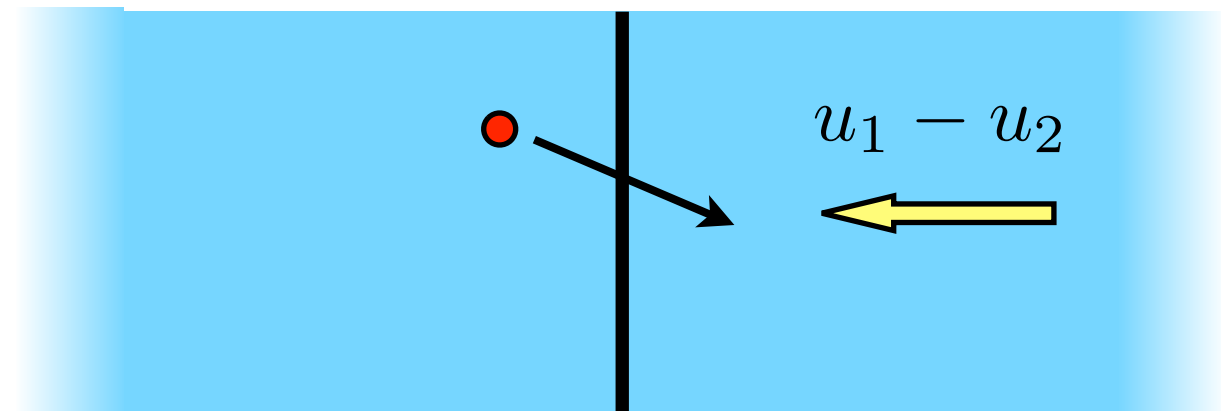
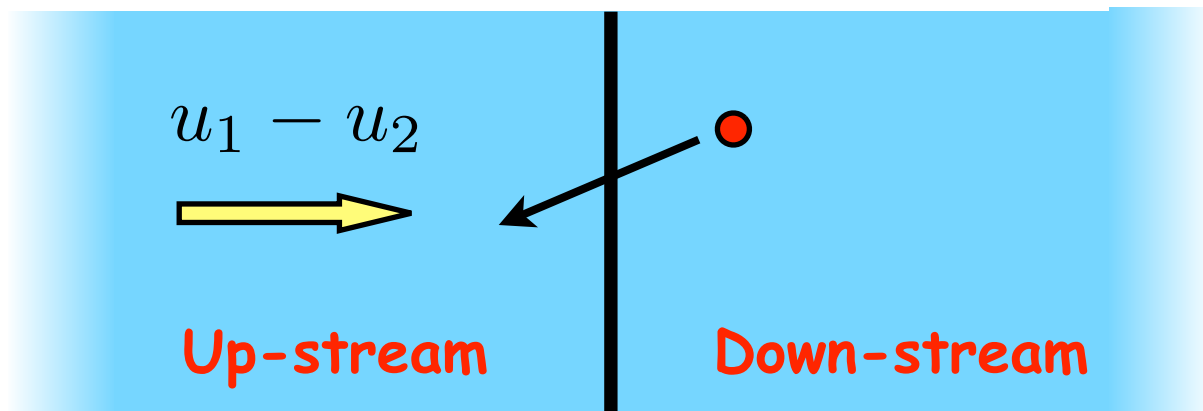
Symmetry



Every time the particle crosses the shock (up $\rightarrow$ down or down $\rightarrow$ up), it undergoes an head-on collision with a plasma moving with velocity $u_1 - u_2$

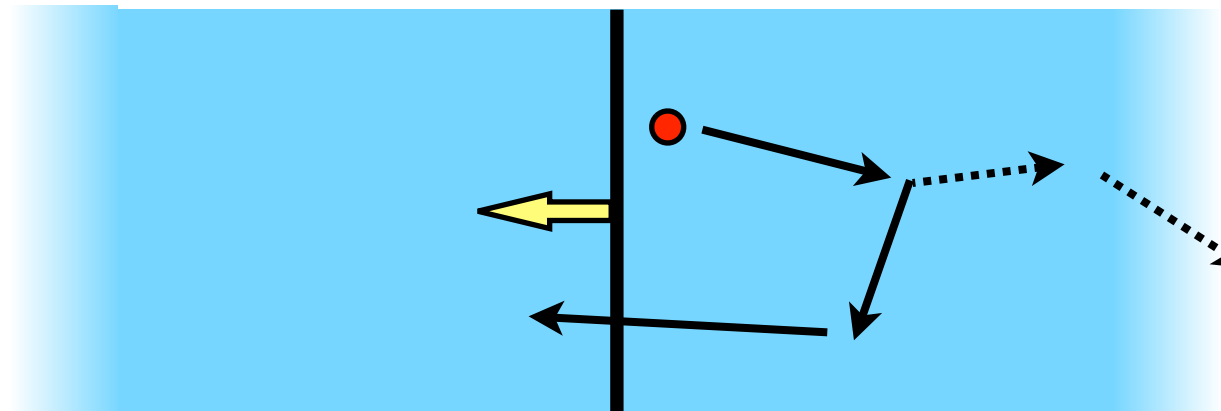
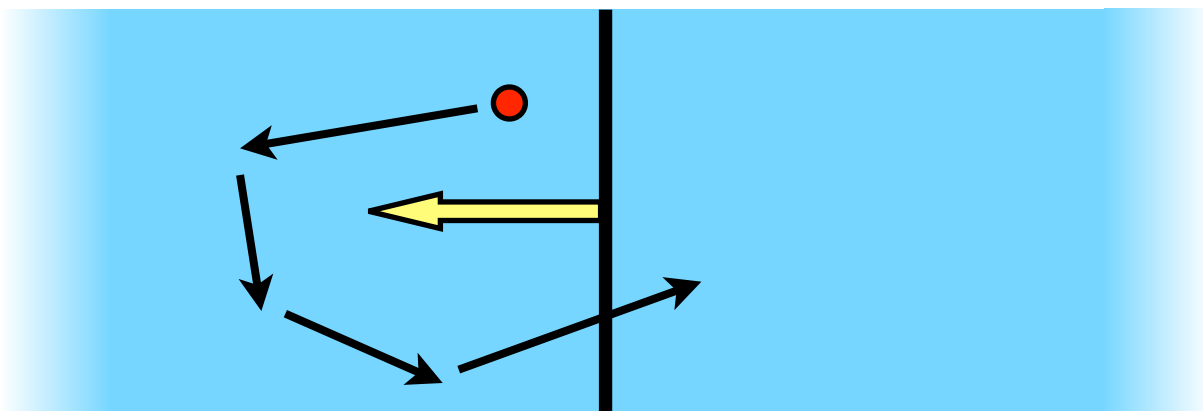
Diffusive Shock Acceleration

Symmetry



Every time the particle crosses the shock (up $\rightarrow$ down or down $\rightarrow$ up), it undergoes a head-on collision with a plasma moving with velocity $u_1 - u_2$

Asymmetry



(Infinite and plane shock:) Upstream particles always return the shock, while downstream particles may be advected and never come back to the shock

Universality of diffusive shock acceleration

Let's search for a test-particle solution

Assumption: scattering is so effective at shocks that the distribution of particles is **very close to isotropy**

-> an universal solution of the problem can be found

$\Delta E/E$ for converging walls

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

$\Delta E/E$ for converging walls

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

averaging over both angles $\rightarrow$

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3} \beta$$

$\Delta E/E$ for converging walls

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

averaging over both angles $\rightarrow$

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3} \beta$$

first order!

$\Delta E/E$ for converging walls

$$\frac{\Delta E}{E} = \beta [\cos(\vartheta'_{out}) - \cos(\vartheta_{in})] + \beta^2 [1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})]$$

averaging over both angles $\rightarrow$

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3} \beta$$

first order!

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3} \beta = \frac{4}{3} \frac{u_1 - u_2}{c} \xrightarrow{u_2 = \frac{u_1}{4}} \frac{u_1}{c}$$

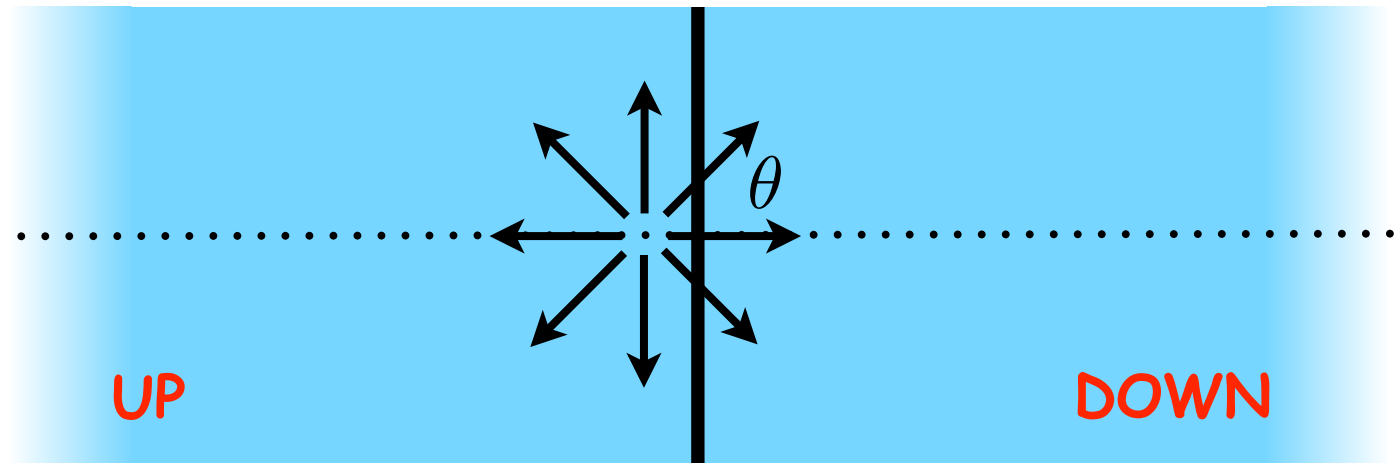
Rate at which particles cross the shock

Let's calculate $R_{in}...$

$n \rightarrow$ density of accelerated particles close to the shock

n is isotropic: $dn = \frac{n}{4\pi} d\Omega$

velocity across the shock: $c \cos(\theta)$



Rate at which particles cross the shock

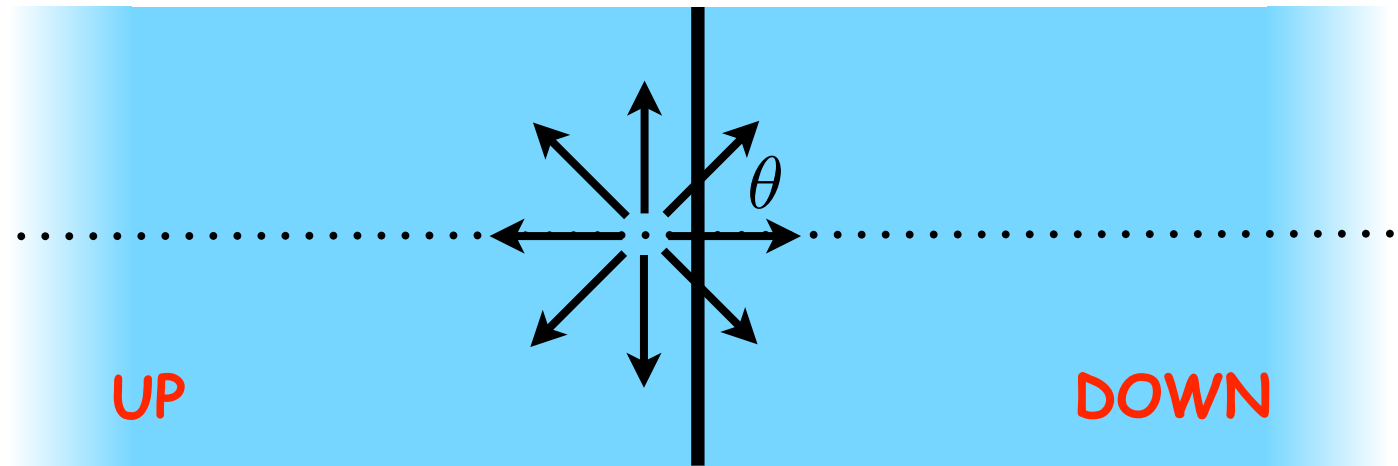
Let's calculate $R_{in}...$

n -> density of accelerated particles close to the shock

n is isotropic: $dn = \frac{n}{4\pi} d\Omega$

velocity across the shock: $c \cos(\theta)$

$$R_{in} = \int_{up \rightarrow down} dn \, c \cos(\theta)$$



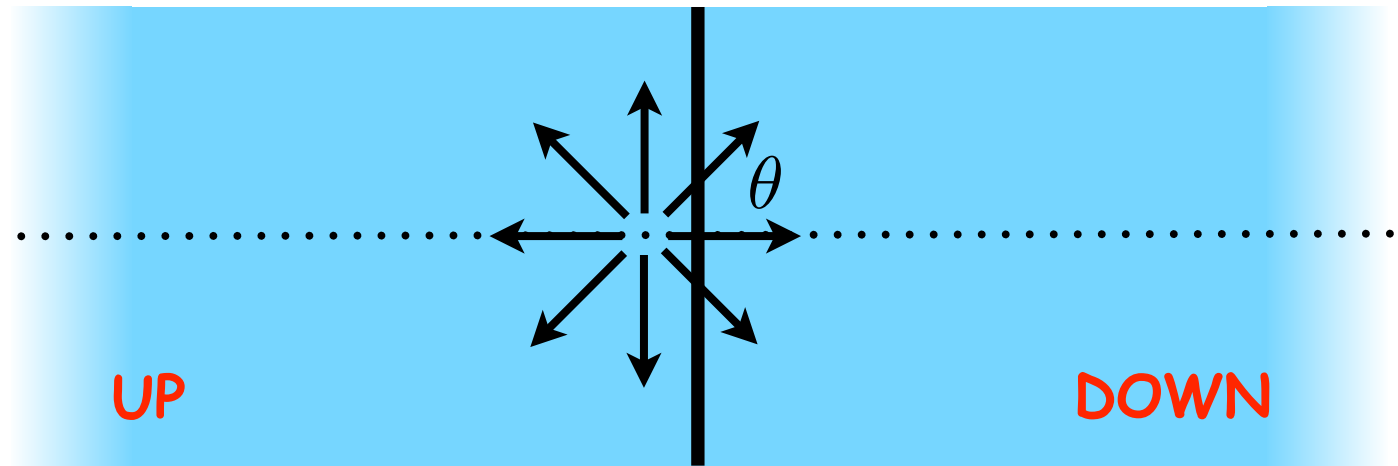
Rate at which particles cross the shock

Let's calculate $R_{in}...$

$n \rightarrow$ density of accelerated particles close to the shock

n is isotropic: $dn = \frac{n}{4\pi} d\Omega$

velocity across the shock: $c \cos(\theta)$



$$R_{in} = \int_{up \rightarrow down} dn \, c \cos(\theta) = \frac{n \, c}{4\pi} \int_0^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta \int_0^{2\pi} d\psi = \frac{1}{4} n \, c$$

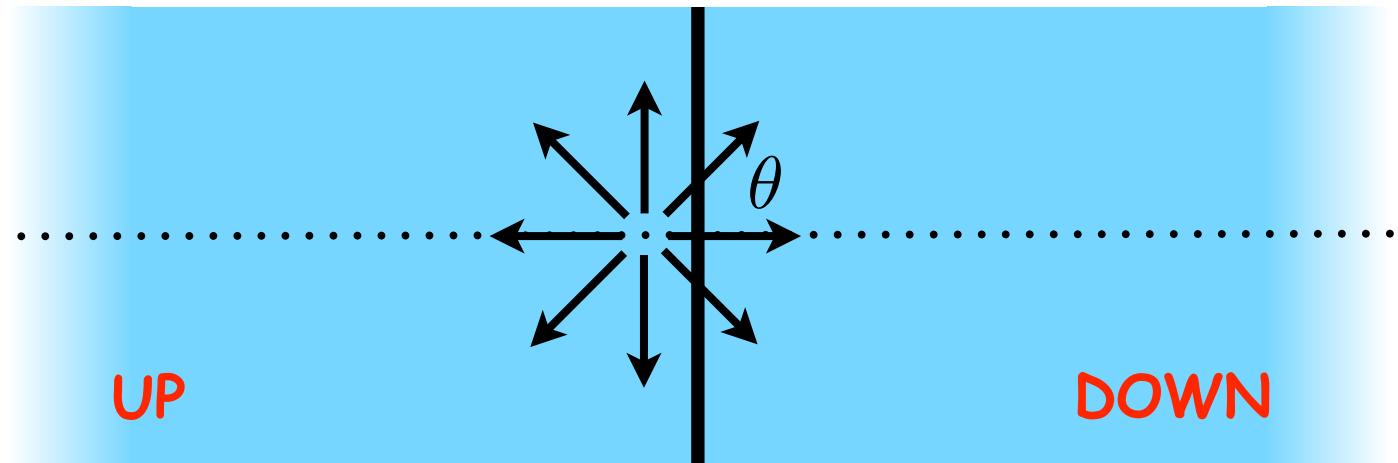
Rate at which particles cross the shock

Let's calculate $R_{in}...$

$n \rightarrow$ density of accelerated particles close to the shock

n is isotropic: $dn = \frac{n}{4\pi} d\Omega$

velocity across the shock: $c \cos(\theta)$

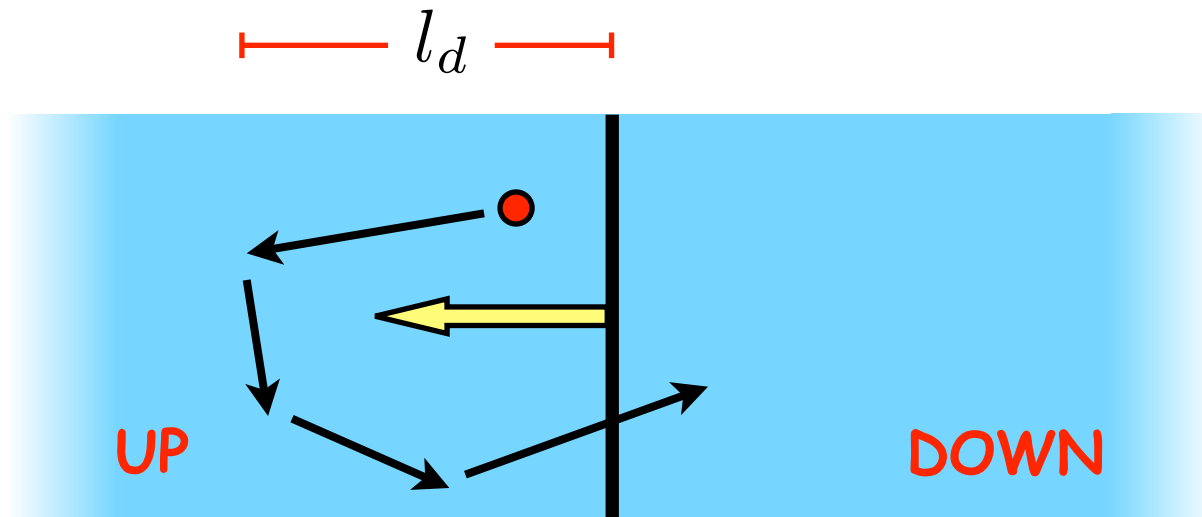


$$R_{in} = \int_{up \rightarrow down} dn \, c \cos(\theta) = \frac{n \, c}{4\pi} \int_0^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta \int_0^{2\pi} d\psi = \frac{1}{4} n \, c$$

$\rightarrow$ the same result is obtained for down $\rightarrow$ up

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



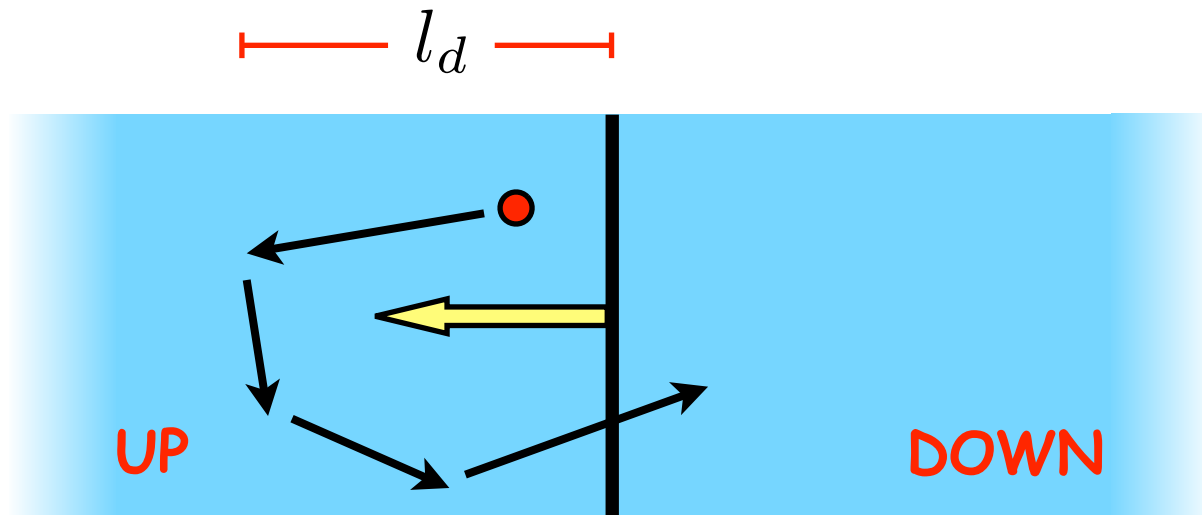
behavior of particles is diffusive

$D(E)$ -> diffusion coefficient

very poorly constrained (from both observations and theory)

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



behavior of particles is diffusive

$D(E)$ -> diffusion coefficient

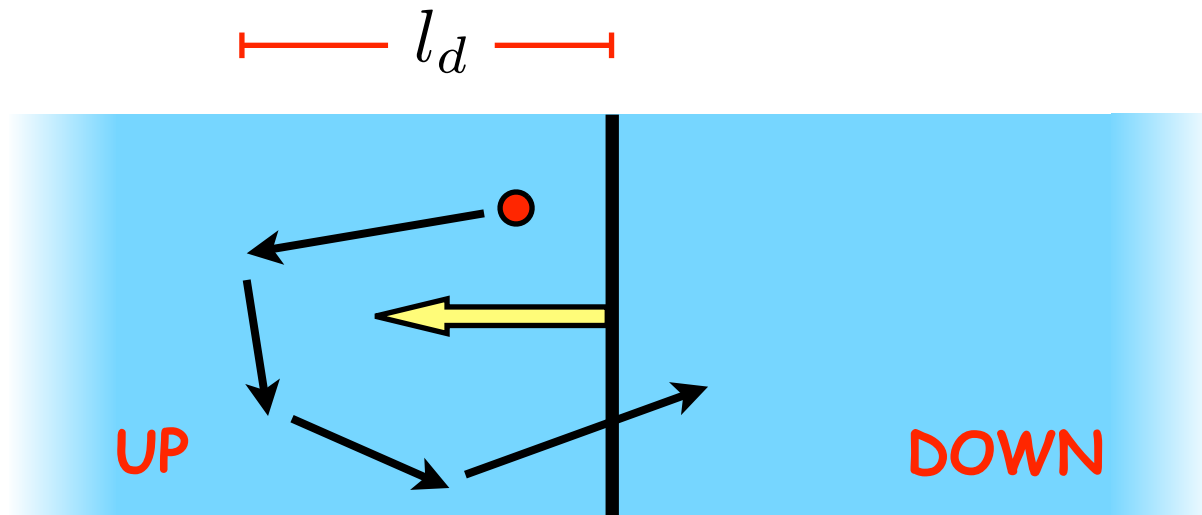
very poorly constrained (from both observations and theory)

-> due to **diffusion** particles spread over

$$l \approx \sqrt{D t}$$

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



behavior of particles is diffusive

$D(E)$ -> diffusion coefficient

very poorly constrained (from both observations and theory)

-> due to **diffusion** particles spread over

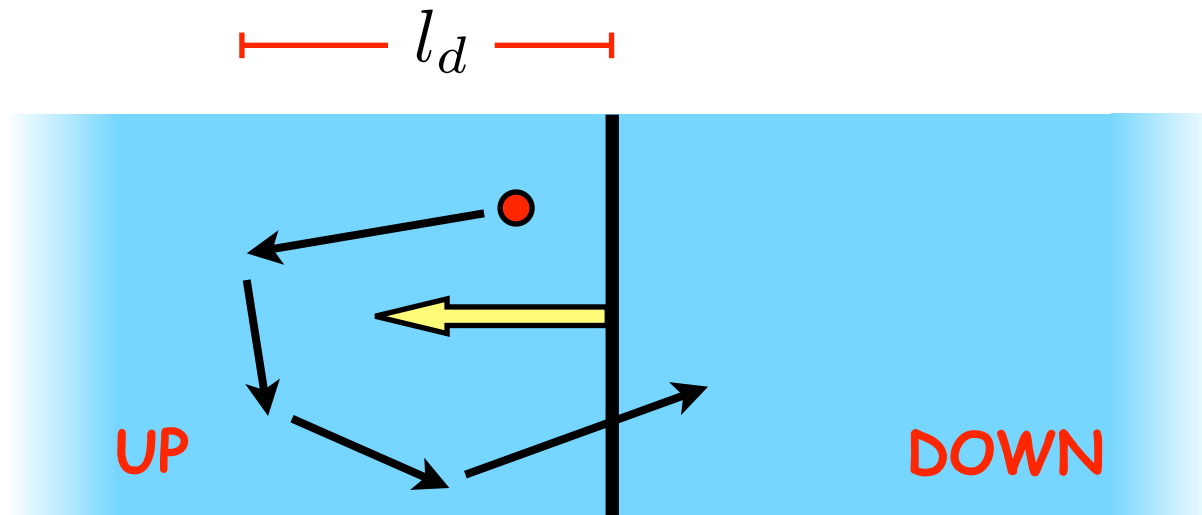
$$l \approx \sqrt{D t}$$

-> at the same time the **shock moves**

$$l = u_1 t$$

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



behavior of particles is diffusive

$D(E)$ -> diffusion coefficient

very poorly constrained (from both observations and theory)

-> due to **diffusion** particles spread over

$$l \approx \sqrt{D t}$$

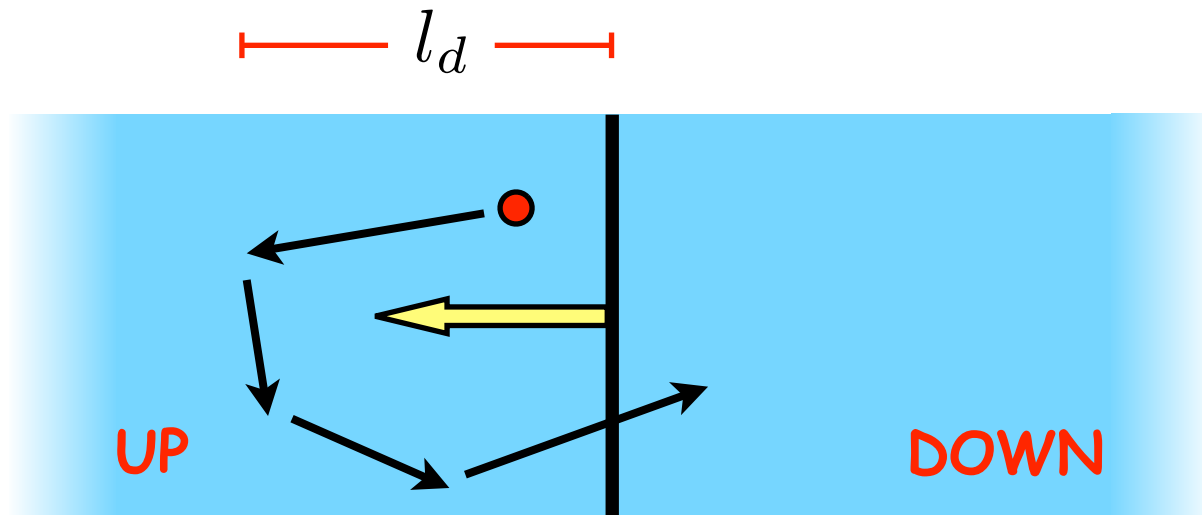
-> at the same time the **shock moves**

$$l = u_1 t$$

$$l_d \approx \frac{D}{u_1}$$

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



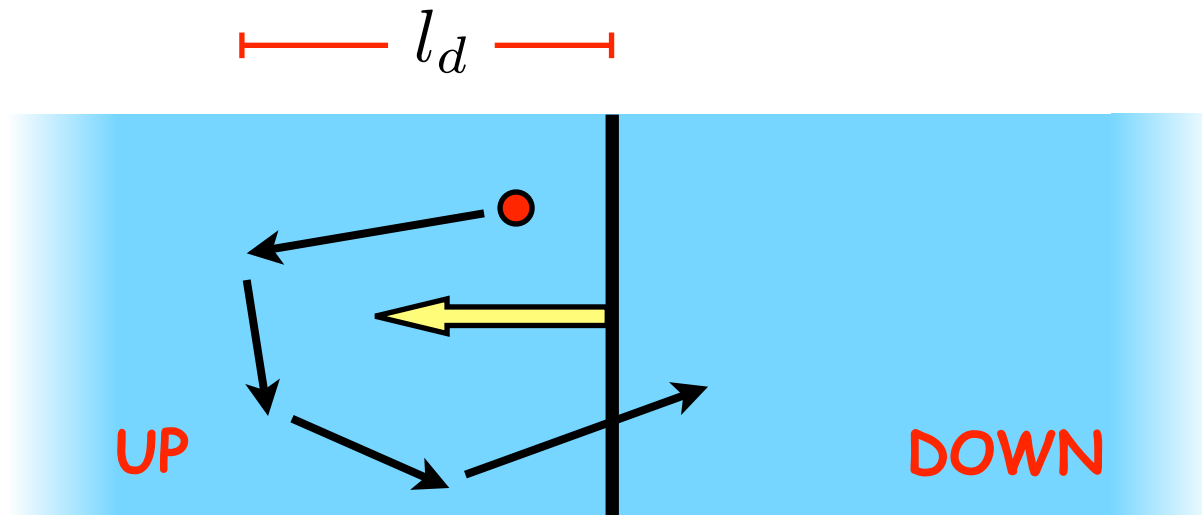
behavior of particles is diffusive

$D(E)$ -> diffusion coefficient

very poorly constrained (from both observations and theory)

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



behavior of particles is diffusive

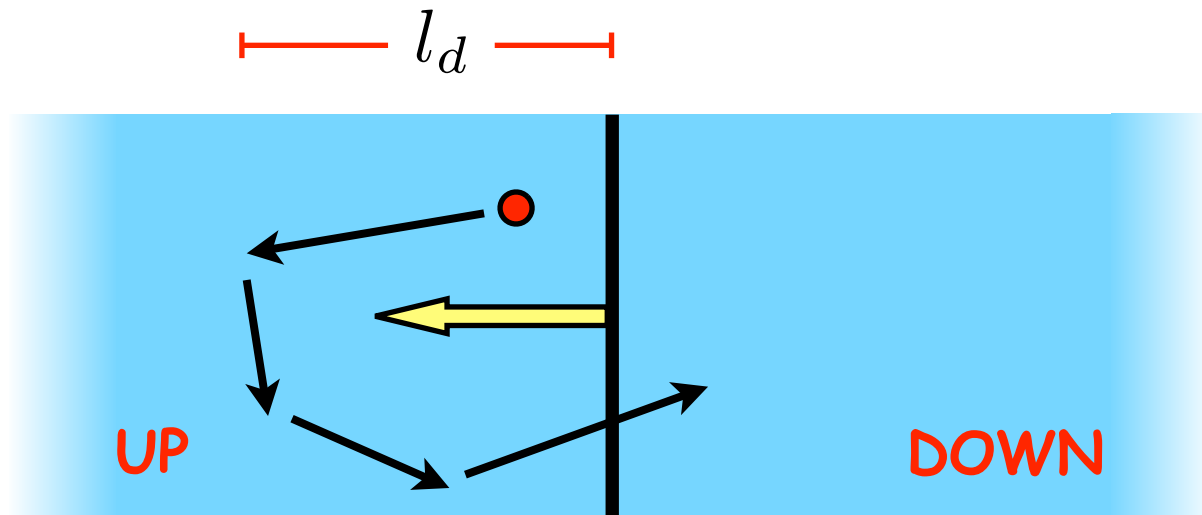
$D(E)$ -> diffusion coefficient

very poorly constrained (from both observations and theory)

cosmic ray precursor -> n ~constant up to $l_d \approx \frac{D}{u_1}$

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



behavior of particles is diffusive

$D(E)$ -> diffusion coefficient

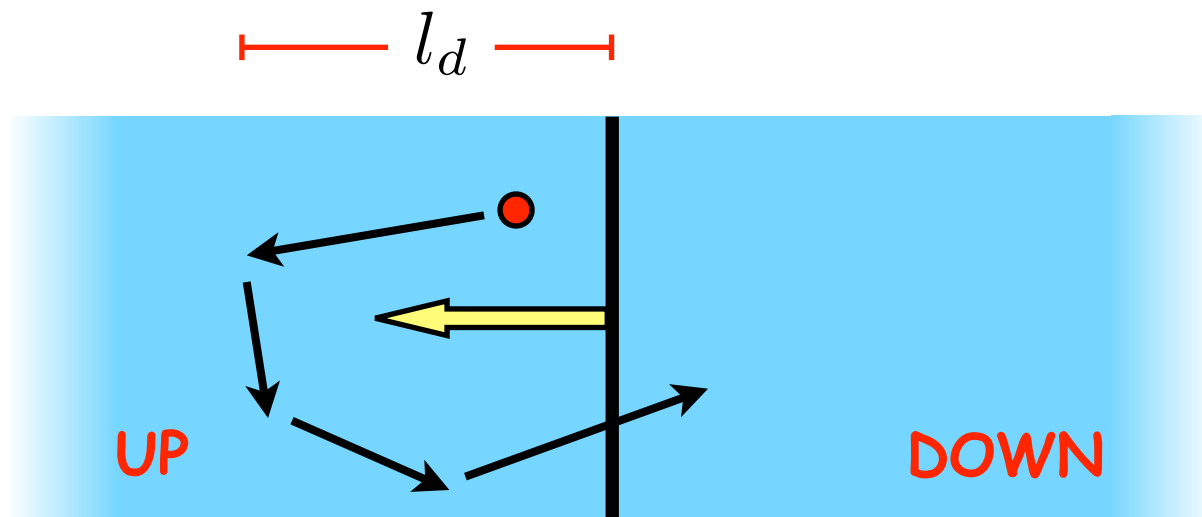
very poorly constrained (from both observations and theory)

cosmic ray precursor -> n ~constant up to $l_d \approx \frac{D}{u_1}$

residence time upstream -> $\tau_{up} = \frac{N_{up}}{R_{in}} = \frac{n l_d}{\frac{1}{4} n c} = \frac{4 D}{u_1 c}$

Residence time upstream

-> let's find the **STEADY STATE** solution upstream of the shock



behavior of particles is diffusive
 $D(E) \rightarrow$ diffusion coefficient

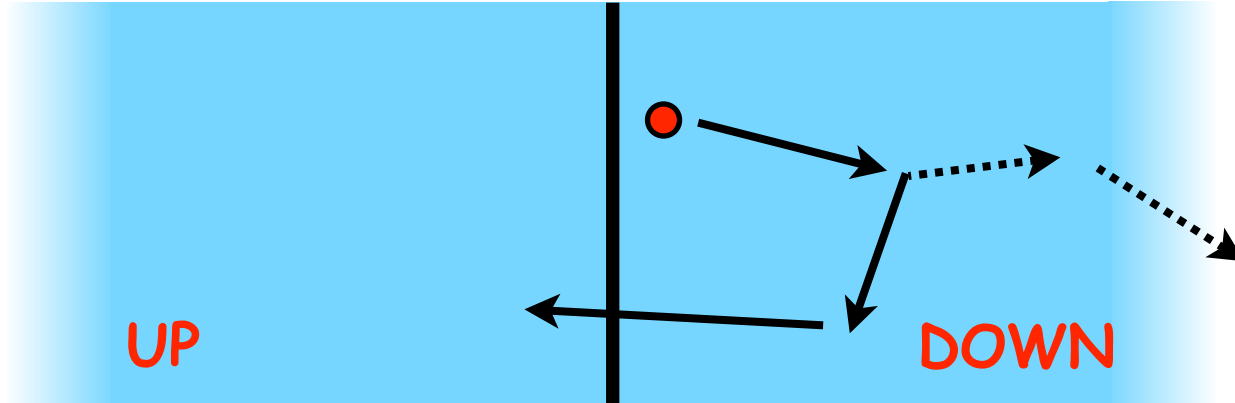
very poorly constrained (from both observations and theory)

cosmic ray precursor -> $n \sim \text{constant up to } l_d \approx \frac{D}{u_1}$

residence time upstream ->
$$\tau_{up} = \frac{N_{up}}{R_{in}} = \frac{n l_d}{\frac{1}{4} n c} = \frac{4 D}{u_1 c}$$

Residence time downstream

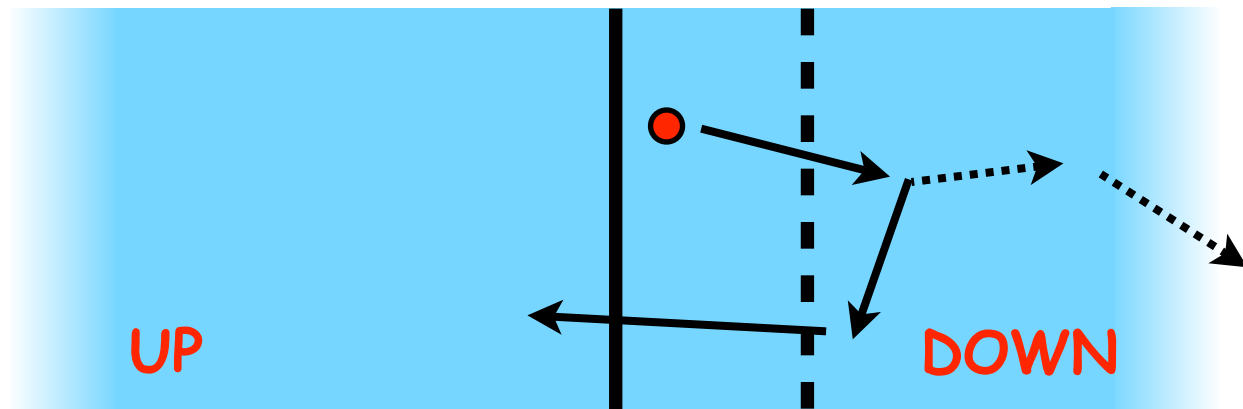
-> a bit more subtle...



n is constant downstream of the shock

Residence time downstream

-> a bit more subtle...



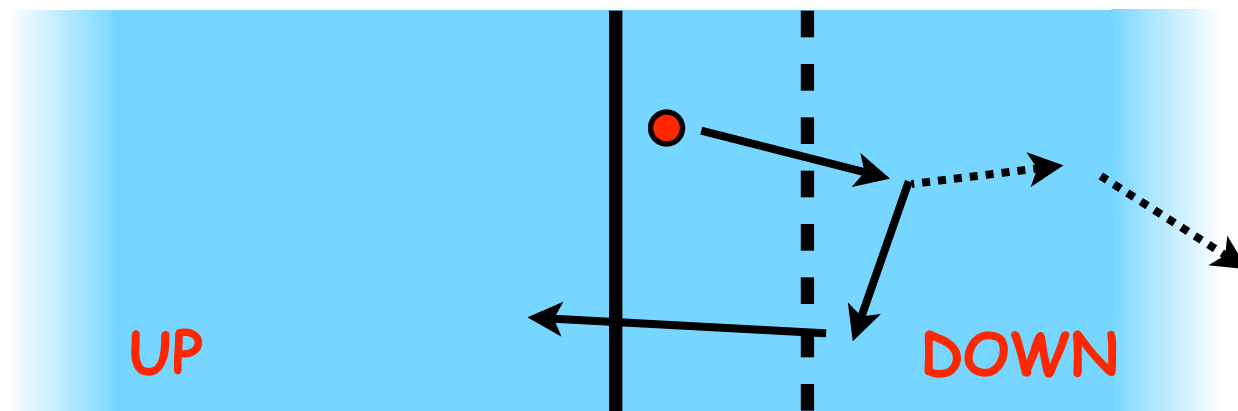
n is constant downstream of the shock

absorbing boundary x_0 source

$$u_2 \frac{\partial n}{\partial x} = D \frac{\partial^2 n}{\partial x^2} + Q \delta(x - x_0) \qquad n(0) = 0$$

Residence time downstream

-> a bit more subtle...



n is constant downstream of the shock

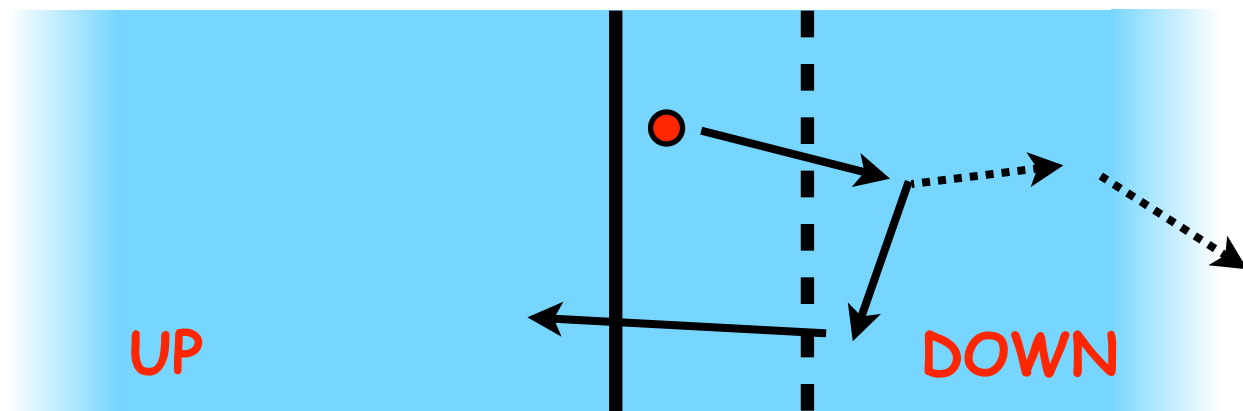
absorbing boundary x_0 source

$$u_2 \frac{\partial n}{\partial x} = D \frac{\partial^2 n}{\partial x^2} + Q \delta(x - x_0) \quad n(0) = 0$$

we need to know the returning flux $D \frac{\partial n}{\partial x} \Big|_{x=0}$

Residence time downstream

-> a bit more subtle...



n is constant downstream of the shock

absorbing boundary x_0 source

$$u_2 \frac{\partial n}{\partial x} = D \frac{\partial^2 n}{\partial x^2} + Q \delta(x - x_0)$$

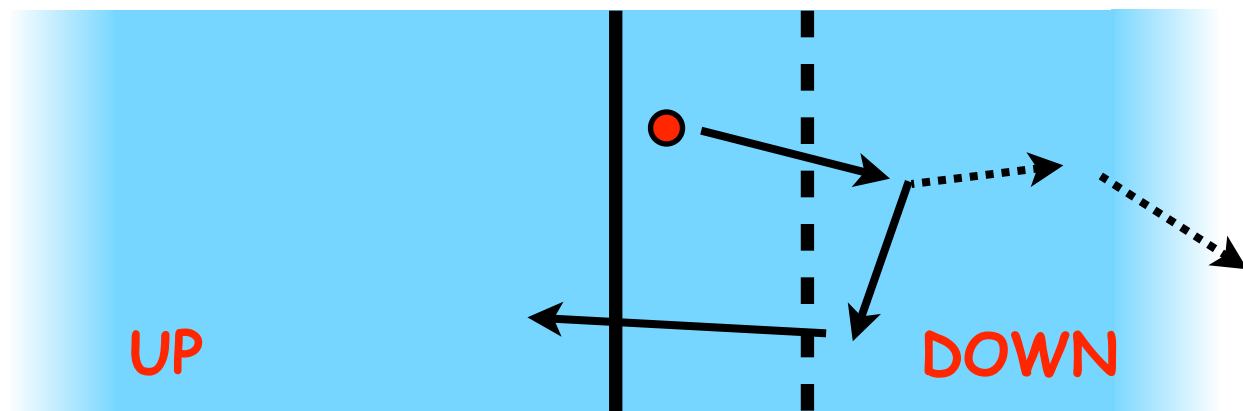
$$n(0) = 0$$

we need to know the returning flux

$$D \frac{\partial n}{\partial x} \Big|_{x=0} \longrightarrow P_{ret} = \frac{D \frac{\partial n}{\partial x} \Big|_{x=0}}{Q}$$

Residence time downstream

-> a bit more subtle...



n is constant downstream of the shock

absorbing boundary x_0 source

$$u_2 \frac{\partial n}{\partial x} = D \frac{\partial^2 n}{\partial x^2} + Q \delta(x - x_0)$$

$$n(0) = 0$$

we need to know the returning flux

$$D \frac{\partial n}{\partial x} \Big|_{x=0}$$

$$\longrightarrow P_{ret} = \frac{D \frac{\partial n}{\partial x} \Big|_{x=0}}{Q}$$

$$P_{ret} = \exp \left(-\frac{x_0 u_2}{D} \right)$$

Residence time downstream

number of downstream particles that will return to the shock:

$$\int_0^\infty dx \, P_{ret}(x) \, n = \frac{D \, n}{u_2} \quad \text{same expression upstream!}$$

Residence time downstream

number of downstream particles that will return to the shock:

$$\int_0^\infty dx P_{ret}(x) n = \frac{D n}{u_2} \quad \text{same expression upstream!}$$

mean residence time upstream $\leftrightarrow$ mean residence time downstream

$$\frac{4D}{u_1 C}$$

$$\frac{4D}{u_2 C}$$

Residence time downstream

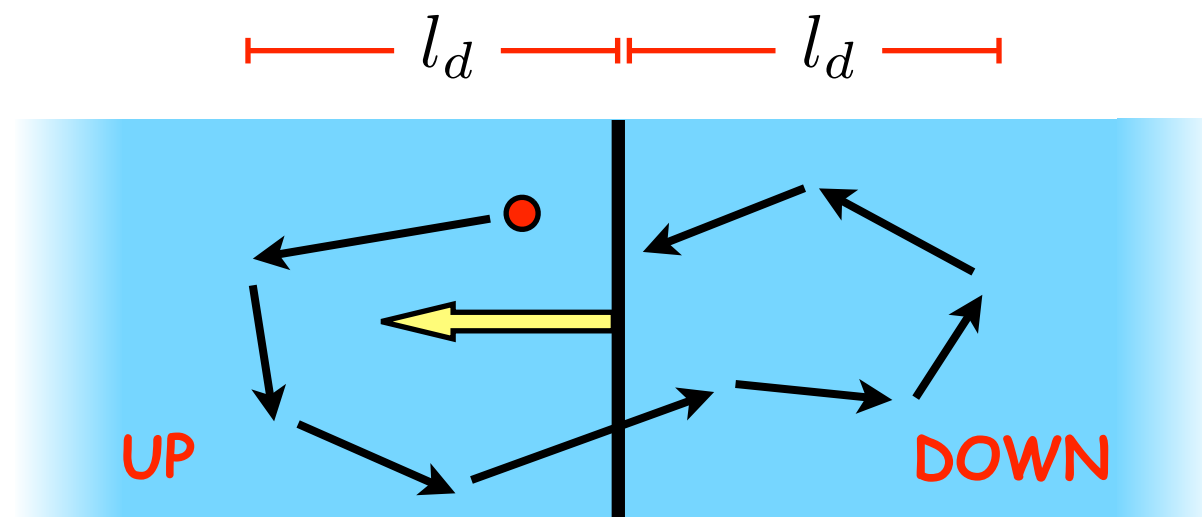
number of downstream particles that will return to the shock:

$$\int_0^\infty dx P_{ret}(x) n = \frac{D n}{u_2} \quad \text{same expression upstream!}$$

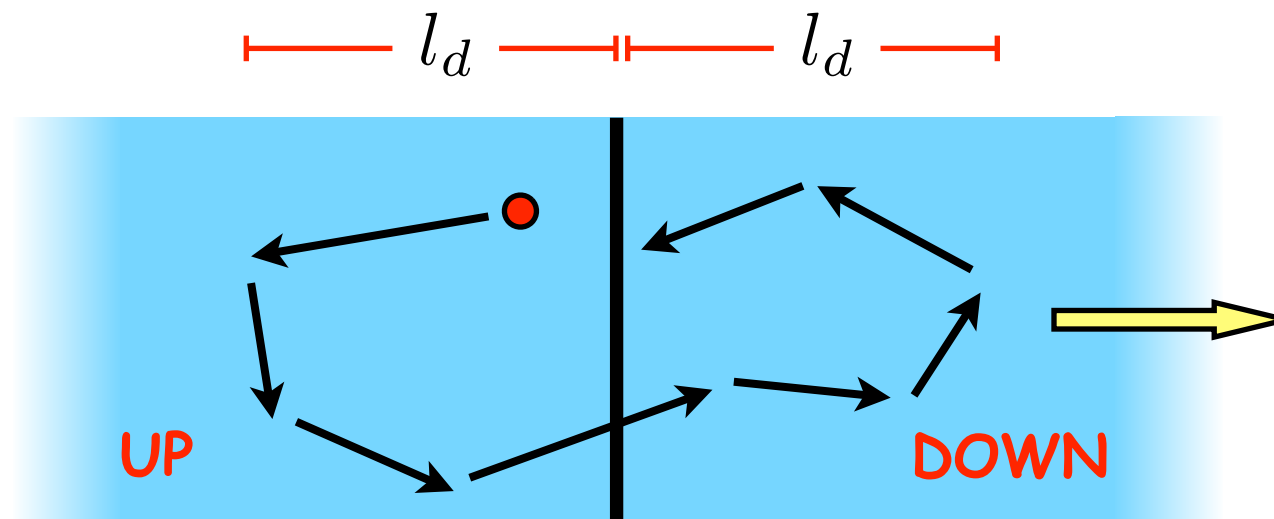
mean residence time upstream $\leftrightarrow$ mean residence time downstream

$$\frac{4D}{u_1 C}$$

$$\frac{4D}{u_2 C}$$



Rate at which particles leave the system



cosmic ray density n is constant downstream...

$$R_{out} = nu_2$$

Acceleration rate

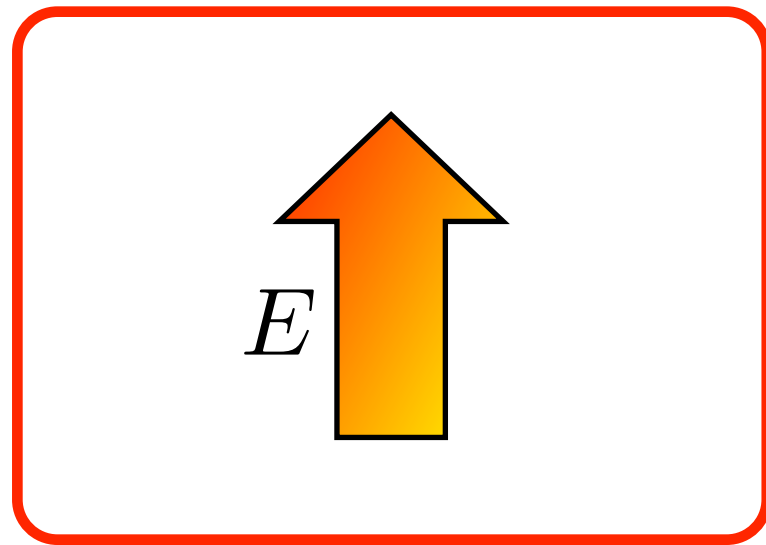


L

everything happens within a "box" of size:

$$L = \frac{D}{u_1} + \frac{D}{u_2} = c \frac{\tau_{up} + \tau_{down}}{4}$$

Acceleration rate

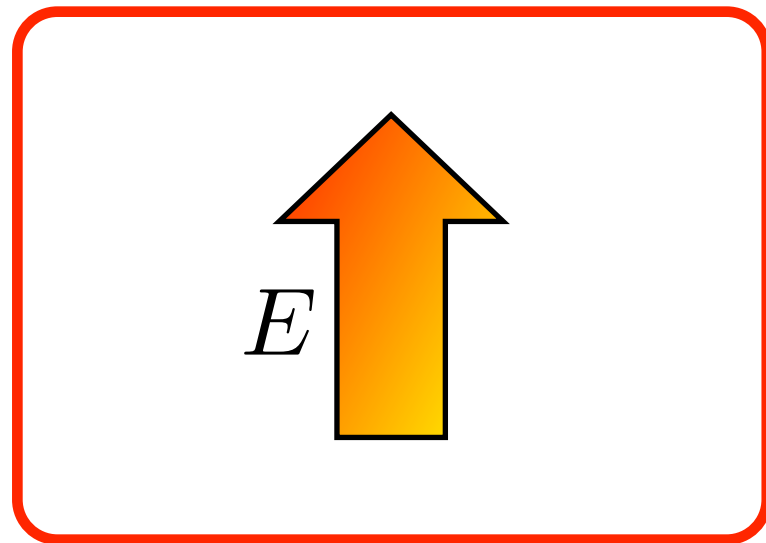


everything happens within a "box" of size:

$$L = \frac{D}{u_1} + \frac{D}{u_2} = c \frac{\tau_{up} + \tau_{down}}{4}$$

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}}$$

Acceleration rate

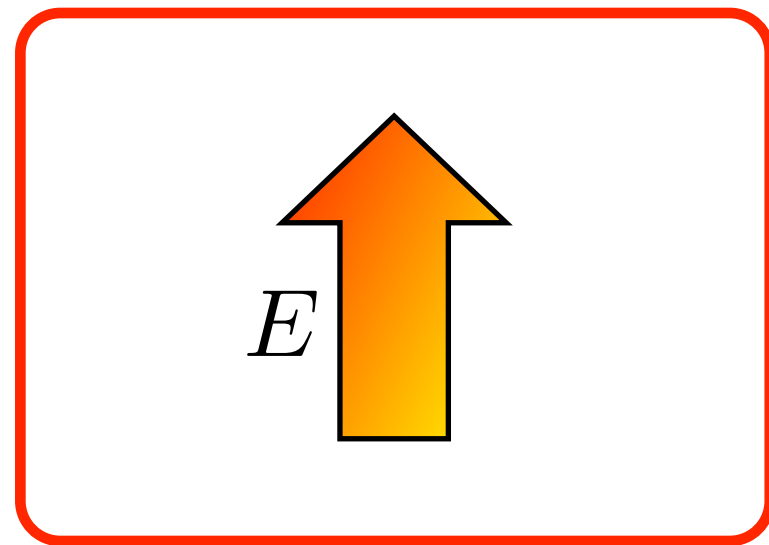


everything happens within a "box" of size:

$$L = \frac{D}{u_1} + \frac{D}{u_2} = c \frac{\tau_{up} + \tau_{down}}{4}$$

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} \approx \frac{\frac{4}{3} \beta c}{4 L} = \boxed{\frac{u_1 - u_2}{3 L}}$$

Box model for shock acceleration

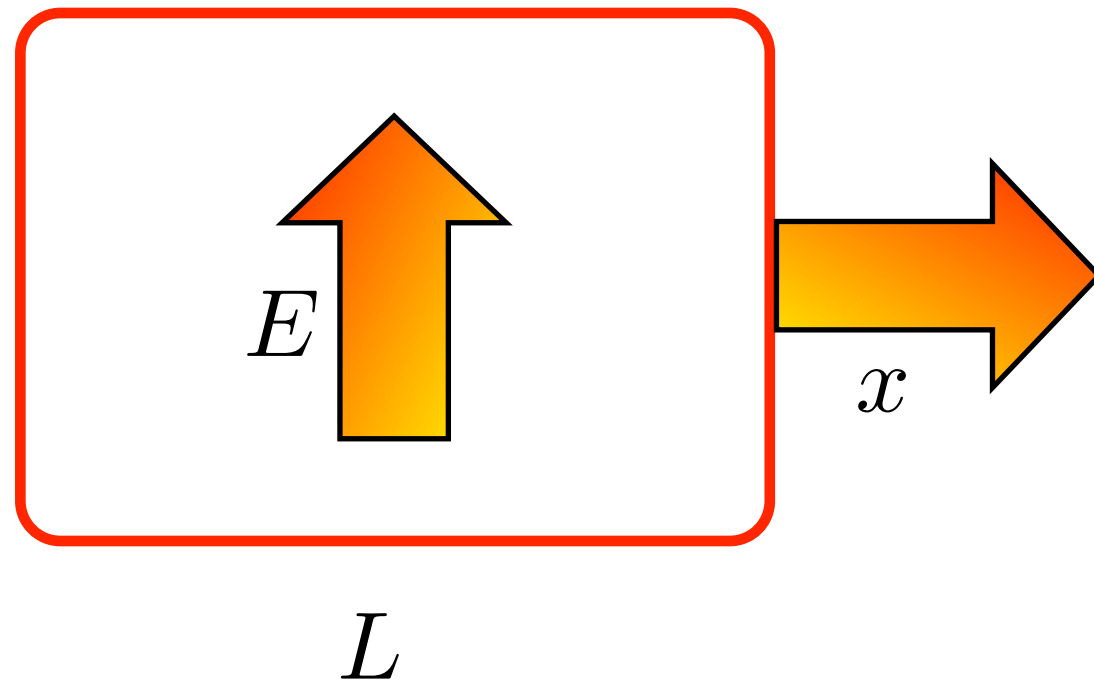


L

particles move up in energy

$$r_{acc} = \frac{u_1 - u_2}{3 L}$$

Box model for shock acceleration



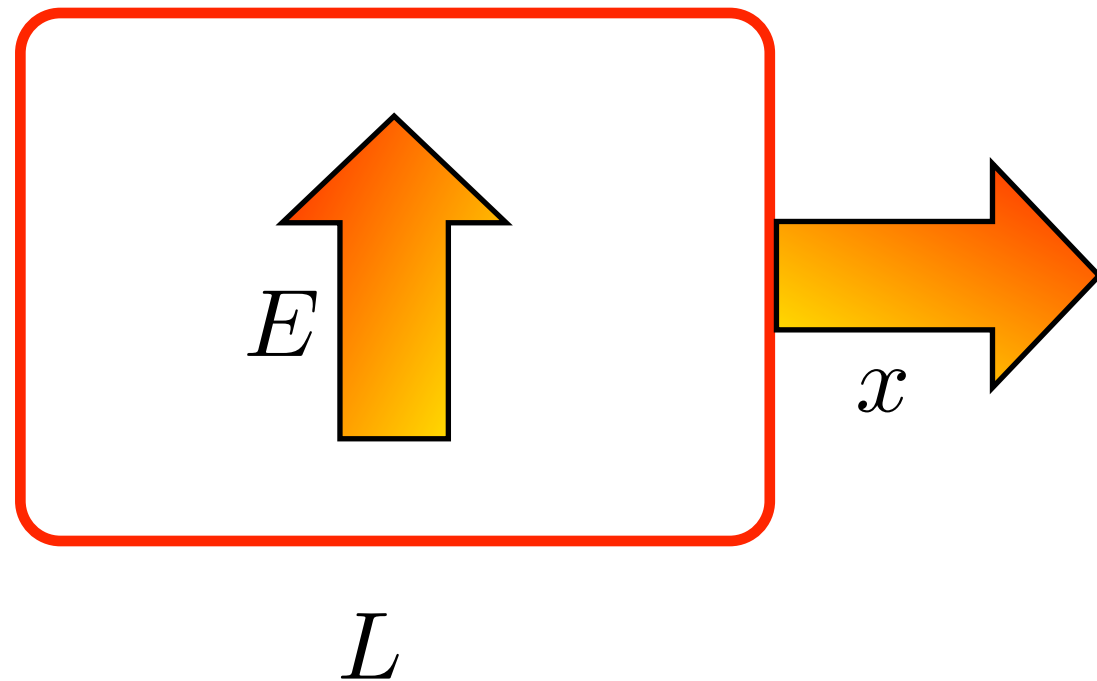
particles move up in energy

$$r_{acc} = \frac{u_1 - u_2}{3 L}$$

particles exit the box downstream

$$r_{esc} = \frac{u_2}{L}$$

Box model for shock acceleration



particles move up in energy

$$r_{acc} = \frac{u_1 - u_2}{3L}$$

particles exit the box downstream

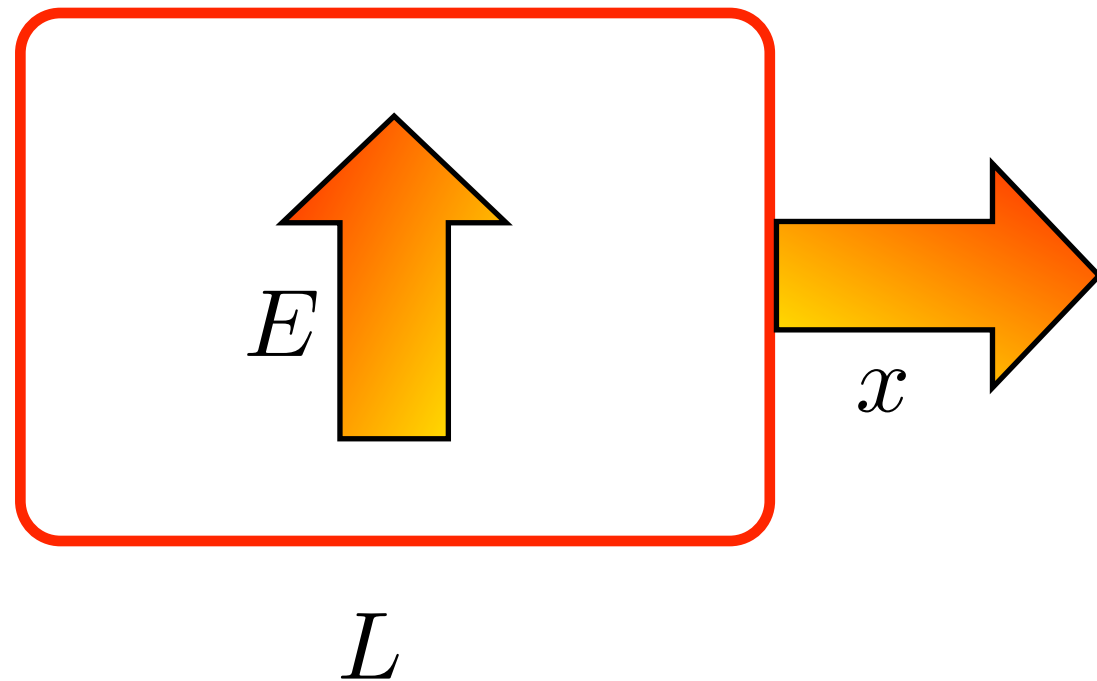
$$r_{esc} = \frac{u_2}{L}$$

up-ward flux in E

down(stream)-ward flux in x

$$\frac{\partial}{\partial E} \left(r_{acc} E N(E) \right) = - r_{esc} N(E)$$

Box model for shock acceleration



particles move up in energy

$$r_{acc} = \frac{u_1 - u_2}{3 L}$$

particles exit the box downstream

$$r_{esc} = \frac{u_2}{L}$$

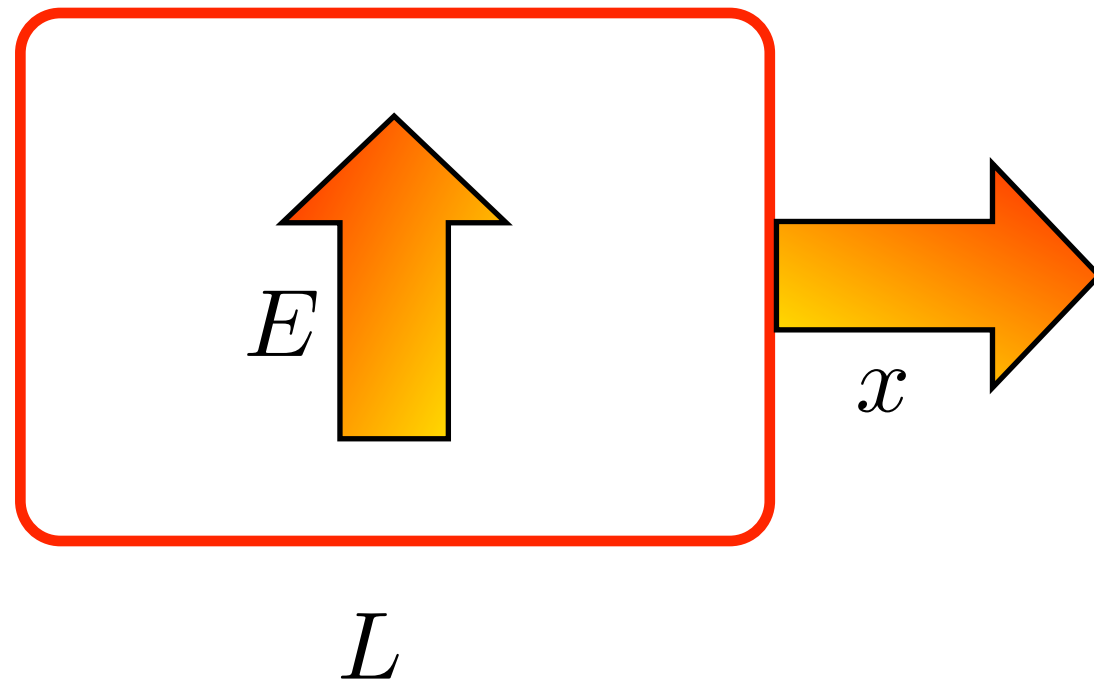
up-ward flux in E

down(stream)-ward flux in x

$$\frac{\partial}{\partial E} \left(r_{acc} E N(E) \right) = - r_{esc} N(E)$$

$$\frac{L}{N(E)} \frac{\partial}{\partial E} \left(E \frac{N(E)}{L} \right) = - \frac{3 u_2}{u_1 - u_2}$$

Box model for shock acceleration



particles move up in energy

$$r_{acc} = \frac{u_1 - u_2}{3 L}$$

particles exit the box downstream

$$r_{esc} = \frac{u_2}{L}$$

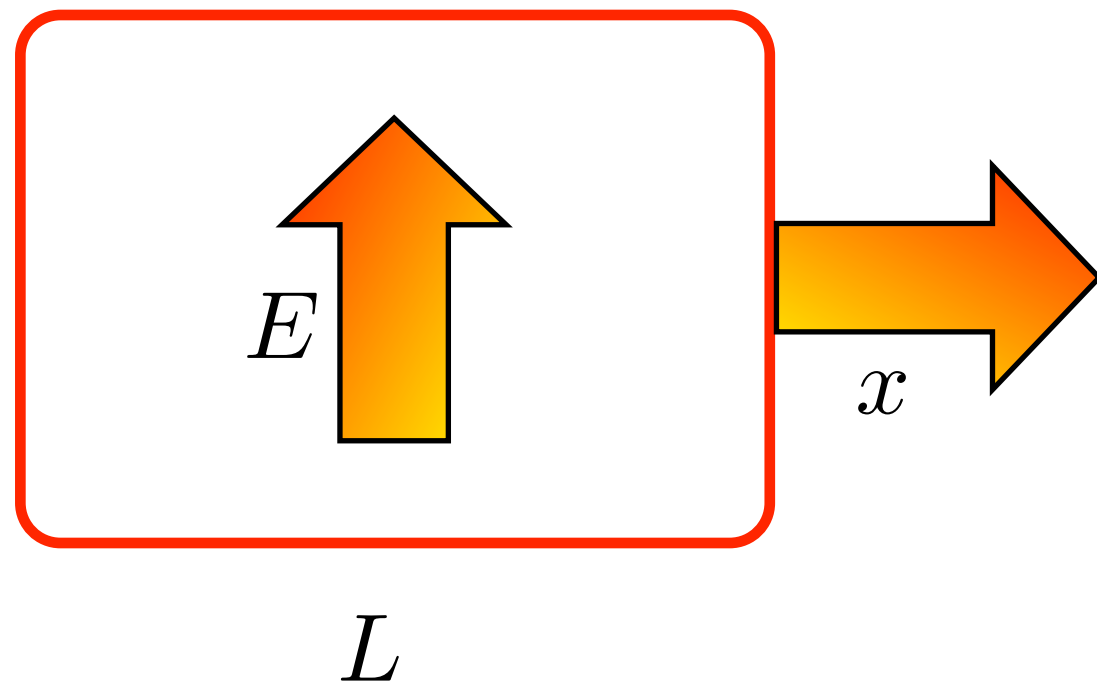
up-ward flux in E

down(stream)-ward flux in x

$$\frac{\partial}{\partial E} \left(r_{acc} E N(E) \right) = - r_{esc} N(E)$$

$$\frac{L}{N(E)} \frac{\partial}{\partial E} \left(E \frac{N(E)}{L} \right) = - \frac{3 u_2}{u_1 - u_2} \quad \leftarrow \begin{matrix} = -1 \end{matrix}$$

Box model for shock acceleration



particles move up in energy

$$r_{acc} = \frac{u_1 - u_2}{3 L}$$

particles exit the box downstream

$$r_{esc} = \frac{u_2}{L}$$

up-ward flux in E

down(stream)-ward flux in x

$$\frac{\partial}{\partial E} \left(r_{acc} E N(E) \right) = - r_{esc} N(E)$$

$$\frac{L}{N(E)} \frac{\partial}{\partial E} \left(E \frac{N(E)}{L} \right) = - \frac{3 u_2}{u_1 - u_2}$$

Annotations for the final equation:

- A red arrow points from $n(E)$ to the boxed term $\frac{N(E)}{L}$.
- A red arrow points from $= -1$ to the coefficient $-$ in front of the fraction on the right-hand side.

Universality of diffusive shock acceleration

Let's search for a test-particle solution

Assumption: scattering is so effective at shocks that
the distribution of particles is **isotropic**

-> an universal solution of the problem can be found

Universality of diffusive shock acceleration

Let's search for a test-particle solution

Assumption: scattering is so effective at shocks that
the distribution of particles is **isotropic**

-> an universal solution of the problem can be found

$$\frac{1}{n(E)} \frac{\partial}{\partial E} \left(E n(E) \right) = -1 \longrightarrow n(E) \propto E^{-2}$$

Universality of diffusive shock acceleration

Let's search for a test-particle solution

Assumption: scattering is so effective at shocks that the distribution of particles is **isotropic**

-> an universal solution of the problem can be found

The diagram illustrates the derivation of the universal solution for diffusive shock acceleration. It shows the continuity equation for particles in a shock, where the diffusion coefficient D is highlighted as being independent of energy. This leads to the power-law distribution $n(E) \propto E^{-2}$.

$$\frac{1}{n} \frac{\partial}{\partial x} \left(D \frac{\partial n}{\partial x} \right) - 1 \longrightarrow n(E) \propto E^{-2}$$

Independent on D !!!

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

-> # of particles starting a cycle per second: $nc/4$

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

-> # of particles starting a cycle per second: $nc/4$

-> # of particles leaving the system per second: $nu_2 = nu_1/4$

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

- > # of particles starting a cycle per second: $nc/4$
- > # of particles leaving the system per second: $nu_2 = nu_1/4$
- > Probability to leave the system per cycle: u_1/c

— divide —>

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

- > # of particles starting a cycle per second: $nc/4$
- > # of particles leaving the system per second: $nu_2 = nu_1/4$
- > Probability to leave the system per cycle: u_1/c
- > Return probability to the shock per cycle: $P_R = 1 - \frac{u_1}{c}$

— divide —>

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

- > # of particles starting a cycle per second: $nc/4$
- > # of particles leaving the system per second: $nu_2 = nu_1/4$
- > Probability to leave the system per cycle: u_1/c
- > Return probability to the shock per cycle: $P_R = 1 - \frac{u_1}{c}$
- > # of particles performing at least k cycles: $N_k = N_0 \left(1 - \frac{u_1}{c}\right)^k$

— divide —
↓

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

-> # of particles starting a cycle per second: $nc/4$

-> # of particles leaving the system per second: $nu_2 = nu_1/4$

— divide —
↓

-> Probability to leave the system per cycle: u_1/c

-> Return probability to the shock per cycle: $P_R = 1 - \frac{u_1}{c}$

-> # of particles performing at least k cycles: $N_k = N_0 \left(1 - \frac{u_1}{c}\right)^k$

-> have an energy larger than: $E_k = E_0 \left(1 + \left\langle \frac{\Delta E}{E} \right\rangle\right)^k = E_0 \left(1 + \frac{u_1}{c}\right)^k$

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0 ...

-> # of particles starting a cycle per second: $nc/4$

-> # of particles leaving the system per second: $nu_2 = nu_1/4$

— divide —
↓

-> Probability to leave the system per cycle: u_1/c

-> Return probability to the shock per cycle: $P_R = 1 - \frac{u_1}{c}$

-> # of particles performing at least k cycles: $N_k = N_0 \left(1 - \frac{u_1}{c}\right)^{\textcircled{k}}$

-> have an energy larger than: $E_k = E_0 \left(1 + \left\langle \frac{\Delta E}{E} \right\rangle\right)^k = E_0 \left(1 + \frac{u_1}{c}\right)^{\textcircled{k}}$

Universal solution: Bell's approach

$$\log \left(\frac{N}{N_0} \right) = k \log \left(1 - \frac{u_1}{c} \right)$$

$$\log \left(\frac{E}{E_0} \right) = k \log \left(1 + \frac{u_1}{c} \right)$$

Universal solution: Bell's approach

$$\log \left(\frac{N}{N_0} \right) = k \log \left(1 - \frac{u_1}{c} \right)$$

$$\log \left(\frac{E}{E_0} \right) = k \log \left(1 + \frac{u_1}{c} \right)$$

— divide —>

$$N(> E) = N_0 \left(\frac{E}{E_0} \right)^{\frac{\log \left(1 - \frac{u_1}{c} \right)}{\log \left(1 + \frac{u_1}{c} \right)}}$$

Universal solution: Bell's approach

$$\log \left(\frac{N}{N_0} \right) = k \log \left(1 - \frac{u_1}{c} \right)$$

$$\log \left(\frac{E}{E_0} \right) = k \log \left(1 + \frac{u_1}{c} \right)$$

— divide —>

$$N(> E) = N_0 \left(\frac{E}{E_0} \right)^{\frac{\log \left(1 - \frac{u_1}{c} \right)}{\log \left(1 + \frac{u_1}{c} \right)}} \longrightarrow -1$$

Universal solution: Bell's approach

$$\log \left(\frac{N}{N_0} \right) = k \log \left(1 - \frac{u_1}{c} \right)$$

$$\log \left(\frac{E}{E_0} \right) = k \log \left(1 + \frac{u_1}{c} \right)$$

— divide —>

$$N(> E) = N_0 \left(\frac{E}{E_0} \right)^{\frac{\log \left(1 - \frac{u_1}{c} \right)}{\log \left(1 + \frac{u_1}{c} \right)}} \longrightarrow -1$$

$$\frac{dN(E)}{dE} \propto E^{-2}$$

Universal solution: Bell's approach

$$\log \left(\frac{N}{N_0} \right) = k \log \left(1 - \frac{u_1}{c} \right)$$

$$\log \left(\frac{E}{E_0} \right) = k \log \left(1 + \frac{u_1}{c} \right)$$

— divide —>

$$N(> E) = N_0 \left(\frac{E}{E_0} \right)^{\frac{\log \left(1 - \frac{u_1}{c} \right)}{\log \left(1 + \frac{u_1}{c} \right)}} \longrightarrow -1$$



where is D?

$$\frac{dN(E)}{dE} \propto E^{-2}$$

Universal solution: Bell's approach

$$\log \left(\frac{N}{N_0} \right) = k \log \left(1 - \frac{u_1}{c} \right)$$

$$\log \left(\frac{E}{E_0} \right) = k \log \left(1 + \frac{u_1}{c} \right)$$

— divide —>

$$N(> E) = N_0 \left(\frac{E}{E_0} \right)^{\frac{\log \left(1 - \frac{u_1}{c} \right)}{\log \left(1 + \frac{u_1}{c} \right)}} \longrightarrow -1$$



where is D?

$$\frac{dN(E)}{dE} \propto E^{-2}$$

UNIVERSAL!!!

Weak shocks

weak $\longrightarrow$ strong

Mach number

$$\mathcal{M} = \frac{u_1}{c_s} \longrightarrow \infty$$

sound speed

Weak shocks

weak $\longrightarrow$ strong

Mach number

$$\mathcal{M} = \frac{u_1}{c_s} \longrightarrow \infty$$

sound speed

$$\frac{Q_2}{Q_1} = \frac{u_1}{u_2} = r \longrightarrow 4$$

$$\frac{dN}{dE} \propto E^{-\frac{3r}{r-1}} \longrightarrow -2$$

How fast?

let's compute the acceleration rate

energy gain
in a cycle

duration
of a cycle

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}}$$

How fast?

let's compute the acceleration rate

energy gain
in a cycle

duration
of a cycle

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} = \frac{\frac{u_1}{c}}{\frac{4D}{u_1 c} + \frac{4D}{u_2 c}}$$

How fast?

let's compute the acceleration rate

energy gain
in a cycle

duration
of a cycle

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} = \frac{\frac{u_1}{c}}{\frac{4D}{u_1 c} + \frac{4D}{u_2 c}} = \frac{u_1^2}{20D}$$

How fast?

let's compute the acceleration rate

energy gain
in a cycle

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} = \frac{\frac{u_1}{c}}{\frac{4D}{u_1 c} + \frac{4D}{u_2 c}} = \frac{u_1^2}{20D}$$

duration
of a cycle

mean free path for
isotropisation

$$D = \frac{1}{3} \lambda c$$

particle velocity

How fast?

let's compute the acceleration rate

energy gain
in a cycle

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} = \frac{\frac{u_1}{c}}{\frac{4D}{u_1 c} + \frac{4D}{u_2 c}} = \frac{u_1^2}{20D}$$

duration
of a cycle

mean free path for
isotropisation

the "elementary"
process is resonant
scattering

$$D = \frac{1}{3} \lambda c \longrightarrow \frac{1}{3} R_L c$$

particle velocity

most optimistic choice for D

How fast?

let's compute the acceleration rate

energy gain in a cycle $\rightarrow \left\langle \frac{\Delta E}{E} \right\rangle$

duration of a cycle $\rightarrow \tau_{up} + \tau_{down}$

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} = \frac{\frac{u_1}{c}}{\frac{4D}{u_1 c} + \frac{4D}{u_2 c}} = \frac{u_1^2}{20D}$$

mean free path for isotropisation $\rightarrow D = \frac{1}{3} \lambda c$

the "elementary" process is resonant scattering $\rightarrow \frac{1}{3} R_L c$

particle velocity $\rightarrow c$

$$D = \frac{1}{3} \lambda c \rightarrow \frac{1}{3} R_L c$$

$$\tau_{acc} = \frac{1}{r_{acc}} = \frac{20 R_L c}{3 u_1^2}$$

most optimistic choice for D

Comparison with Hillas

Hillas acceleration time →

$$\tau_{acc}^H = \left(\frac{v}{c} \right)^{-1} \frac{R_L}{c}$$

Comparison with Hillas

Hillas acceleration time $\rightarrow$

$$\tau_{acc}^H = \left(\frac{v}{c} \right)^{-1} \frac{R_L}{c}$$

Fermi II $\rightarrow$

$$\tau_{acc} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c} \right)^{-1} \tau_{acc}^H$$

Comparison with Hillas

Hillas acceleration time $\rightarrow$

$$\tau_{acc}^H = \left(\frac{v}{c}\right)^{-1} \frac{R_L}{c}$$

Fermi II $\rightarrow$

$$\tau_{acc} = \alpha^{-1} \frac{\lambda}{R_L} \left(\frac{v}{c}\right)^{-1} \tau_{acc}^H$$

shock acceleration $\rightarrow$

$$\tau_{acc} = a \left(\frac{v}{c}\right)^{-1} \tau_{acc}^H$$

Comparison with Hillas

Hillas acceleration time →

$$\tau_{acc}^H = \left(\frac{v}{c}\right)^{-1} \frac{R_L}{c}$$

Fermi II →

$$\tau_{acc} = \alpha^{-1} \left(\frac{\lambda}{R_L}\right) \left(\frac{v}{c}\right)^{-1} \tau_{acc}^H$$

shock acceleration →

$$\tau_{acc} = a \left(\frac{v}{c}\right)^{-1} \tau_{acc}^H$$

factor of
several

$\gg 1$

DSA is faster than Fermi II but still (obviously) slower than Hillas

Particle acceleration at relativistic shocks

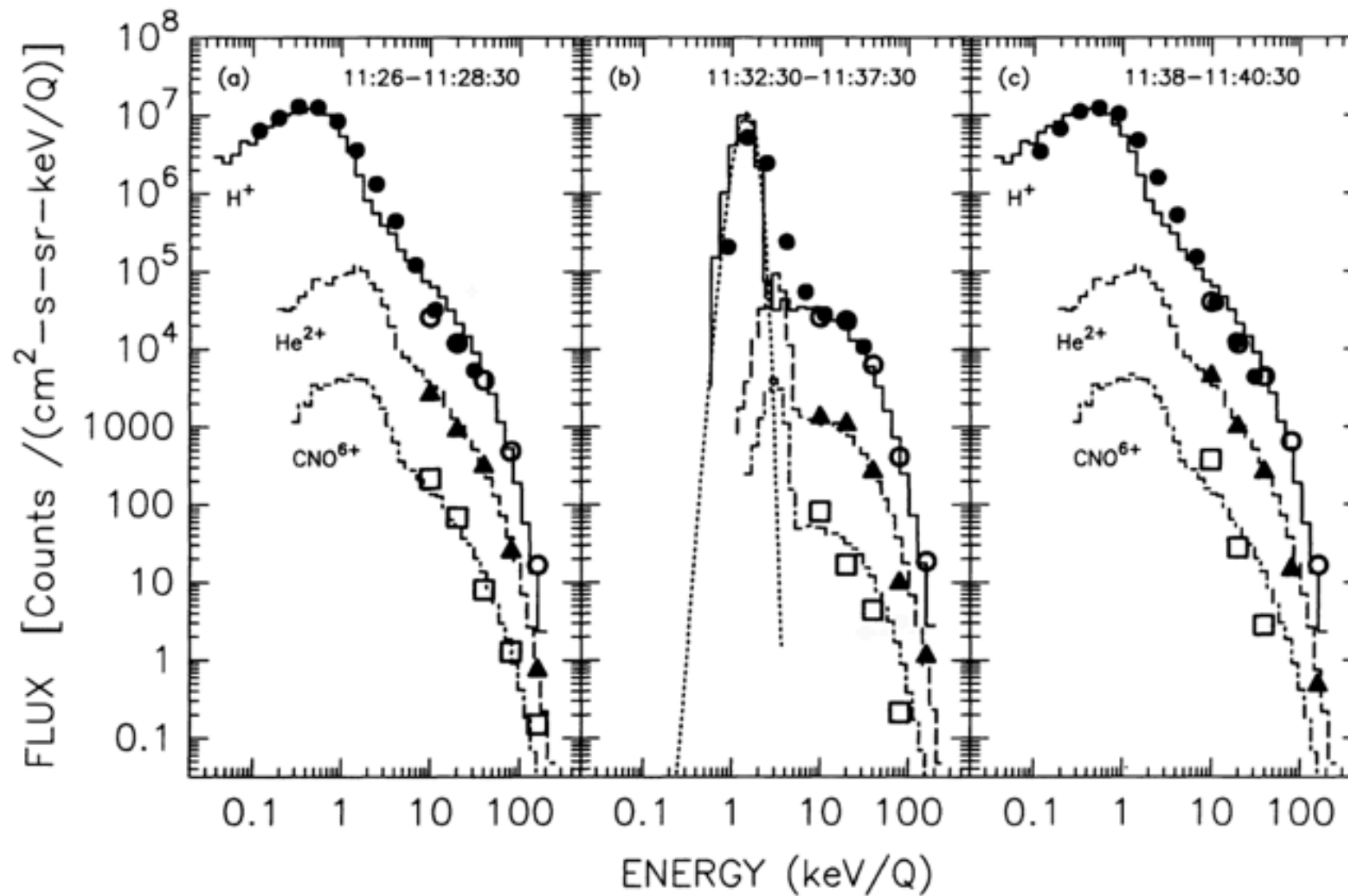
Achterberg et al. (2001)

Achterberg, Lecture notes, Les Houches School (2004)

**Acceleration @non-relativistic/relativistic
shocks: most important difference**

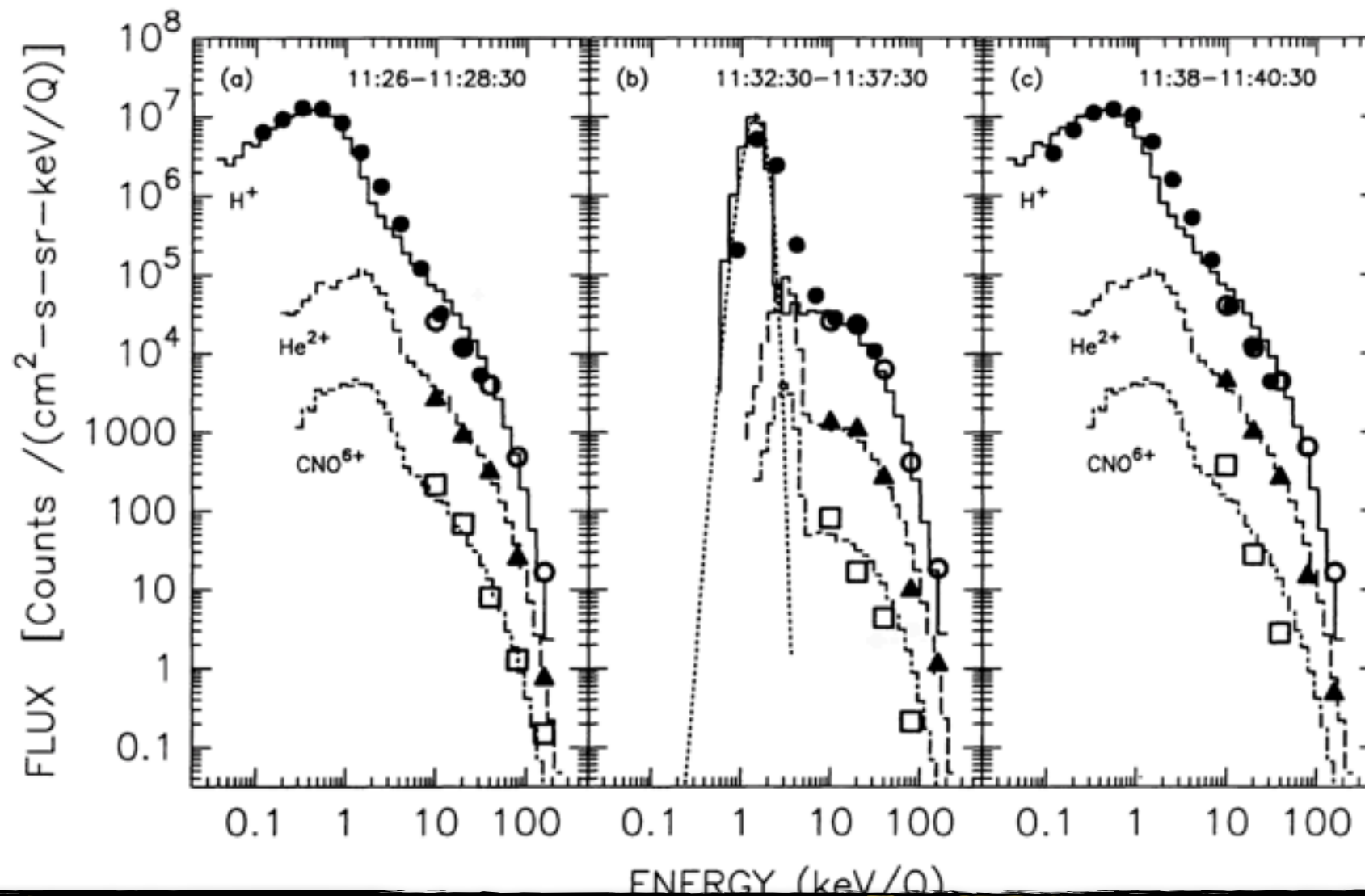
Acceleration @non-relativistic/relativistic shocks: most important difference

DSA at non relativistic shock EXISTS!



Acceleration @non-relativistic/relativistic shocks: most important difference

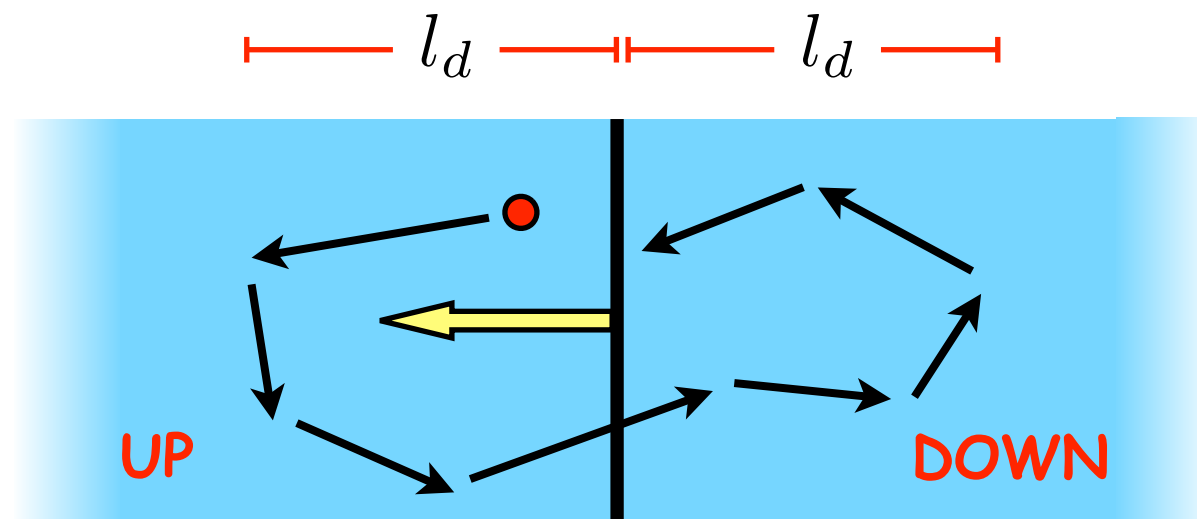
DSA at non relativistic shock EXISTS!



space probes "saw" it! (down-up-downstream passage at the Earth bow shock)

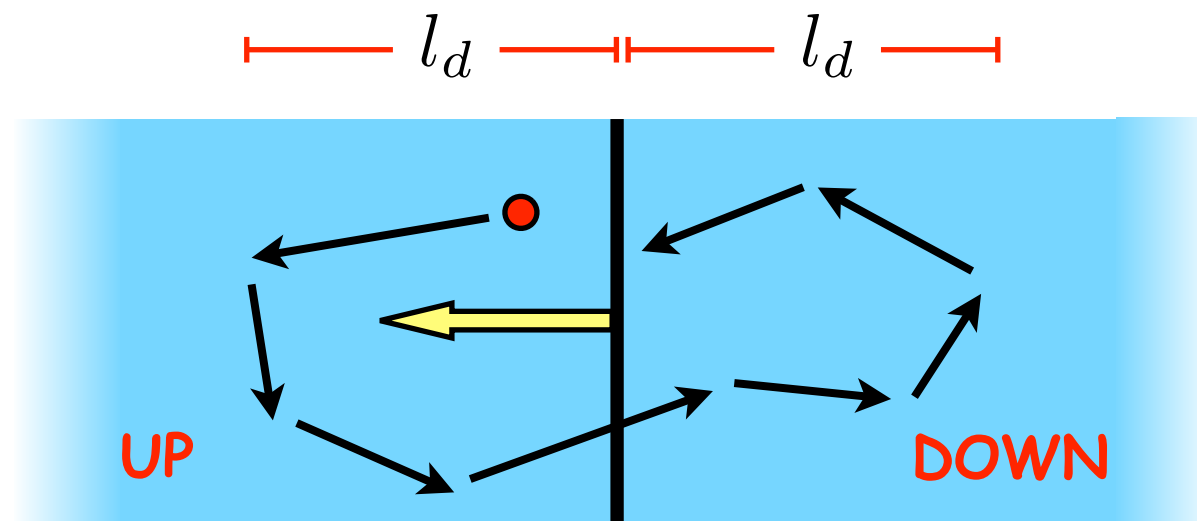
Key aspects of DSA at non-relativistic shocks

particles are accelerated through a series of cycles up—>down—>up stream



Key aspects of DSA at non-relativistic shocks

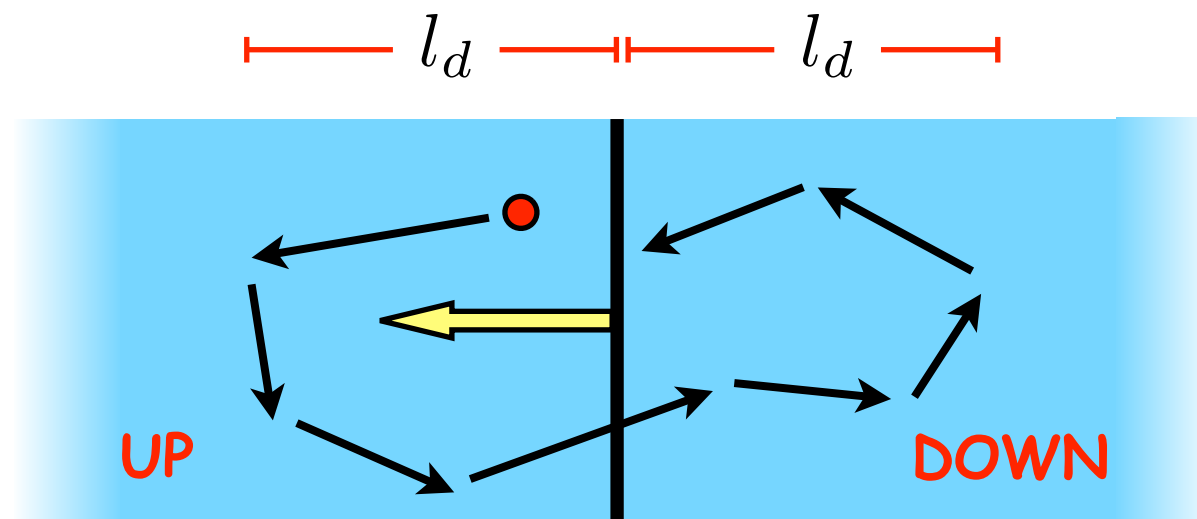
particles are accelerated through a series of cycles up→down→up stream



■ diffusive transport → isotropy

Key aspects of DSA at non-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream



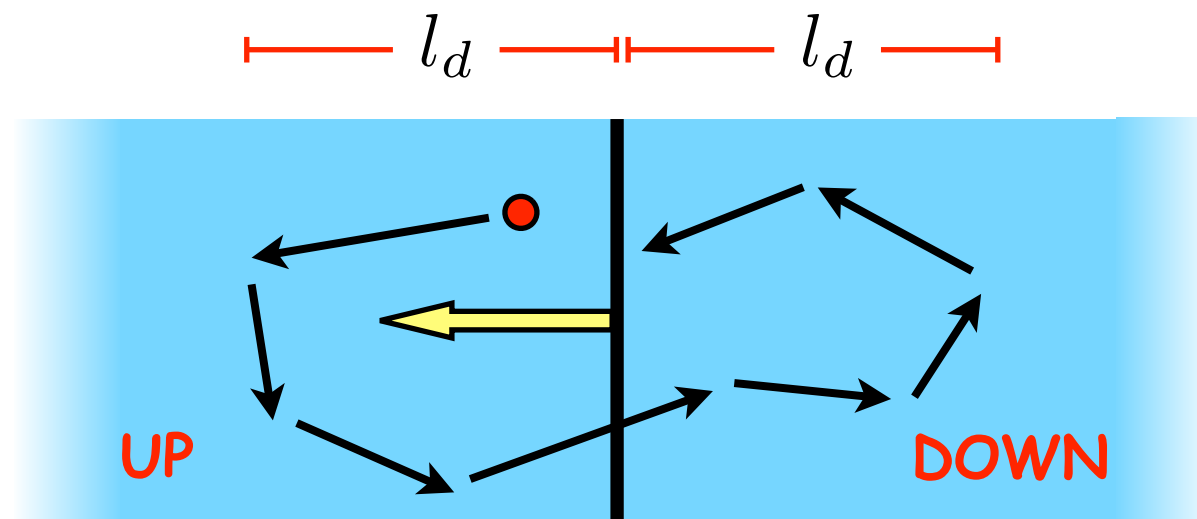
■ diffusive transport → **isotropy**

■ energy gain per cycle → **small**

$$\Delta_{acc} = \Delta E / E = u_1 / c$$

Key aspects of DSA at non-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream



■ diffusive transport → **isotropy**

■ energy gain per cycle → **small**

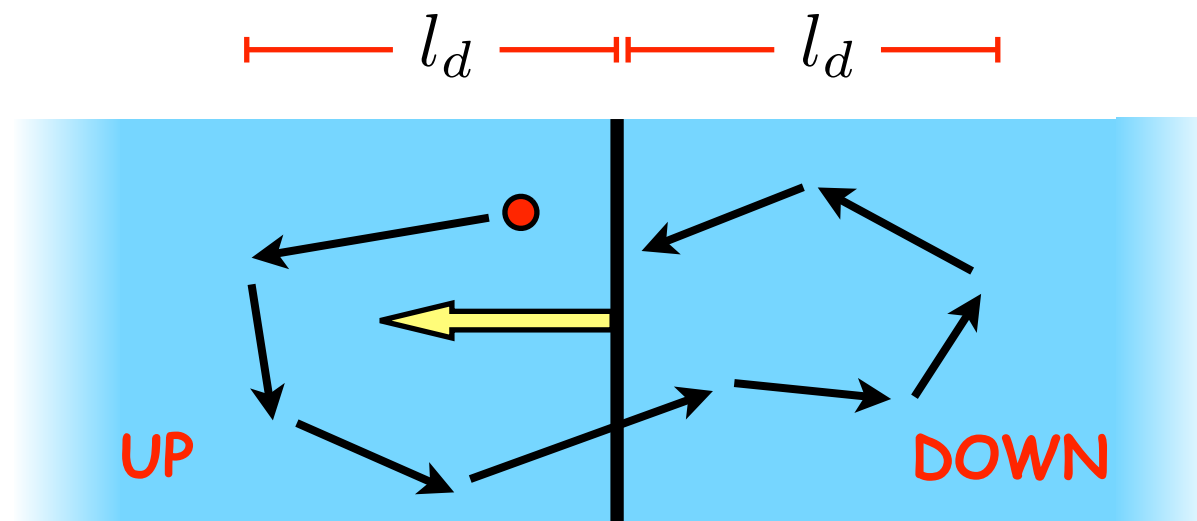
$$\Delta_{acc} = \Delta E / E = u_1 / c$$

■ escape probability per cycle → **small**

$$P_{esc} = u_1 / c$$

Key aspects of DSA at non-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream



- diffusive transport → **isotropy**
- energy gain per cycle → **small**
- escape probability per cycle → **small**
- spectral slope → **E^{-s}**

$$\Delta_{acc} = \Delta E / E = u_1 / c$$

$$P_{esc} = u_1 / c$$

$$s = 1 + \frac{P_{esc}}{\Delta_{acc}}$$

What happens if we consider a relativistic shock

$c=1$

The velocity of the shock is comparable
to the velocity of accelerated particles

What happens if we consider a relativistic shock

$c=1$

The velocity of the shock is comparable
to the velocity of accelerated particles

consider an ultra-relativistic shock $\Gamma_s \gg 1$

What happens if we consider a relativistic shock

$c=1$

The velocity of the shock is comparable
to the velocity of accelerated particles

consider an ultra-relativistic shock $\Gamma_s \gg 1$

$$\Gamma_s = \frac{1}{(1 - \beta_s)^{1/2}} \longrightarrow \beta_s = \left(1 - \frac{1}{\Gamma_s^2}\right)^{1/2} \approx 1 - \frac{1}{2\Gamma_s^2}$$

What happens if we consider a relativistic shock

$c=1$

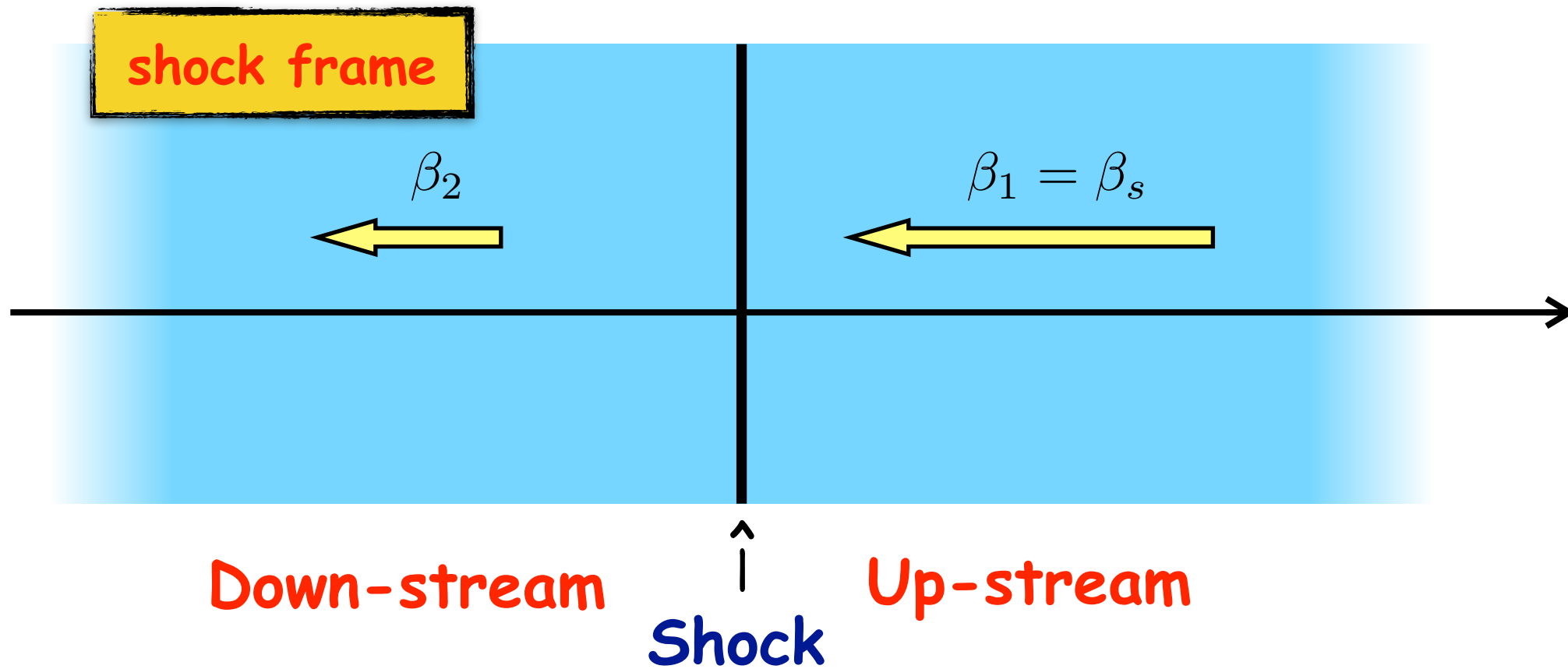
The velocity of the shock is comparable
to the velocity of accelerated particles

consider an ultra-relativistic shock $\Gamma_s \gg 1$

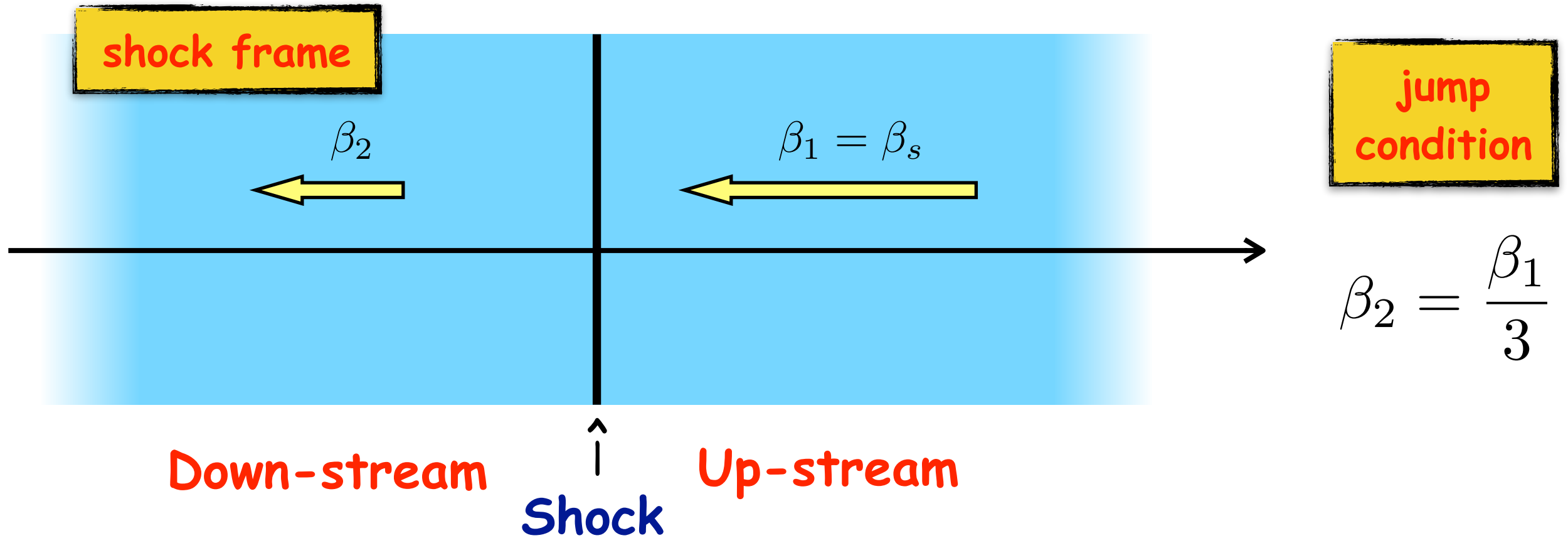
$$\Gamma_s = \frac{1}{(1 - \beta_s)^{1/2}} \longrightarrow \beta_s = \left(1 - \frac{1}{\Gamma_s^2}\right)^{1/2} \approx 1 - \frac{1}{2\Gamma_s^2}$$

$$\beta_s \approx 1 - \frac{1}{2\Gamma_s^2}$$

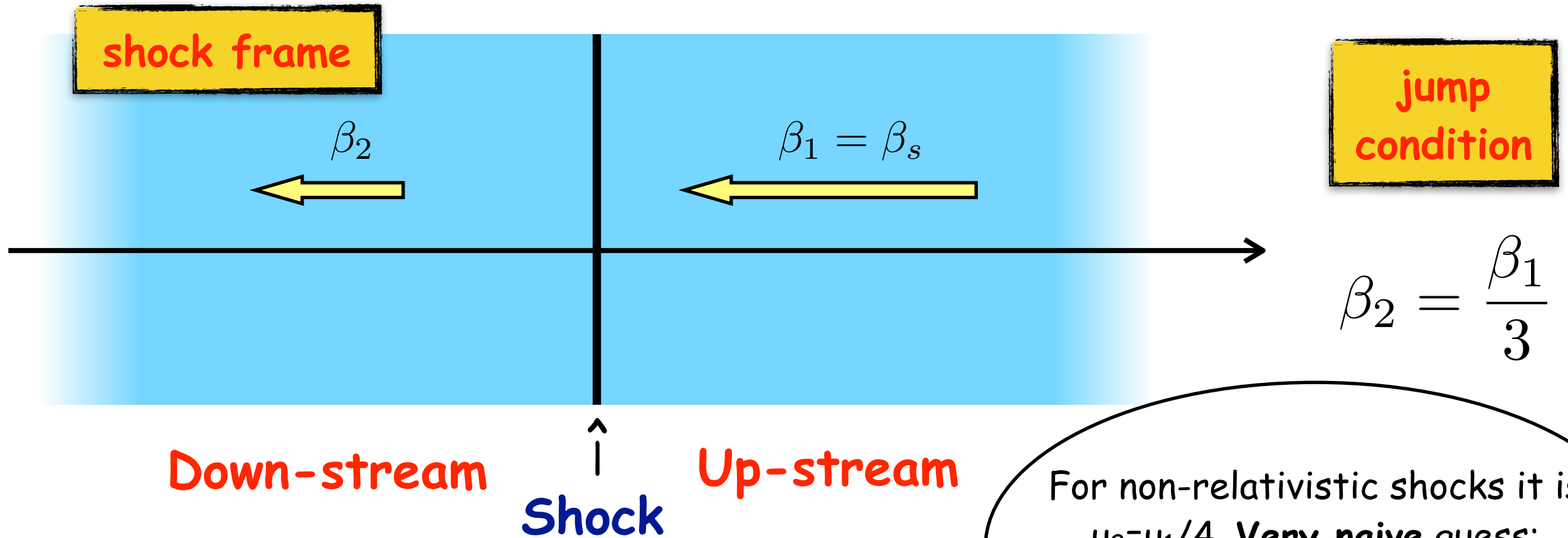
Ultra-relativistic shocks



Ultra-relativistic shocks



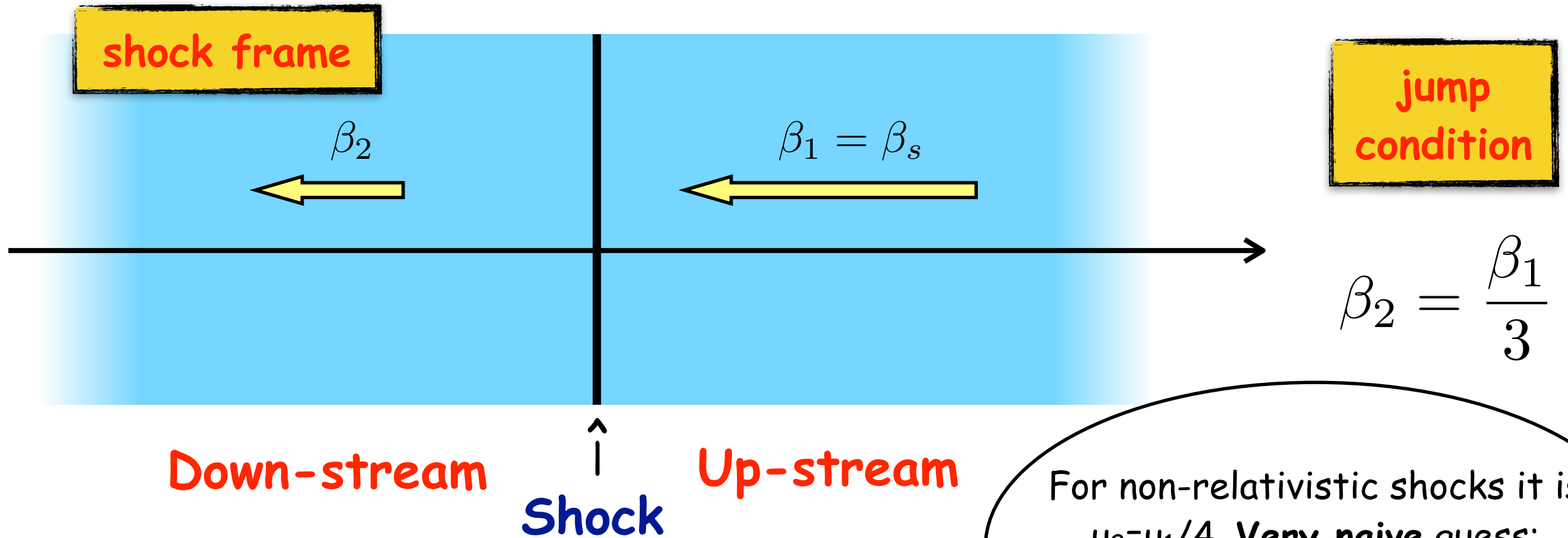
Ultra-relativistic shocks



For non-relativistic shocks it is $u_2 = u_1/4$. Very naive guess: spectra are in this case steeper than E^{-2} ?



Ultra-relativistic shocks



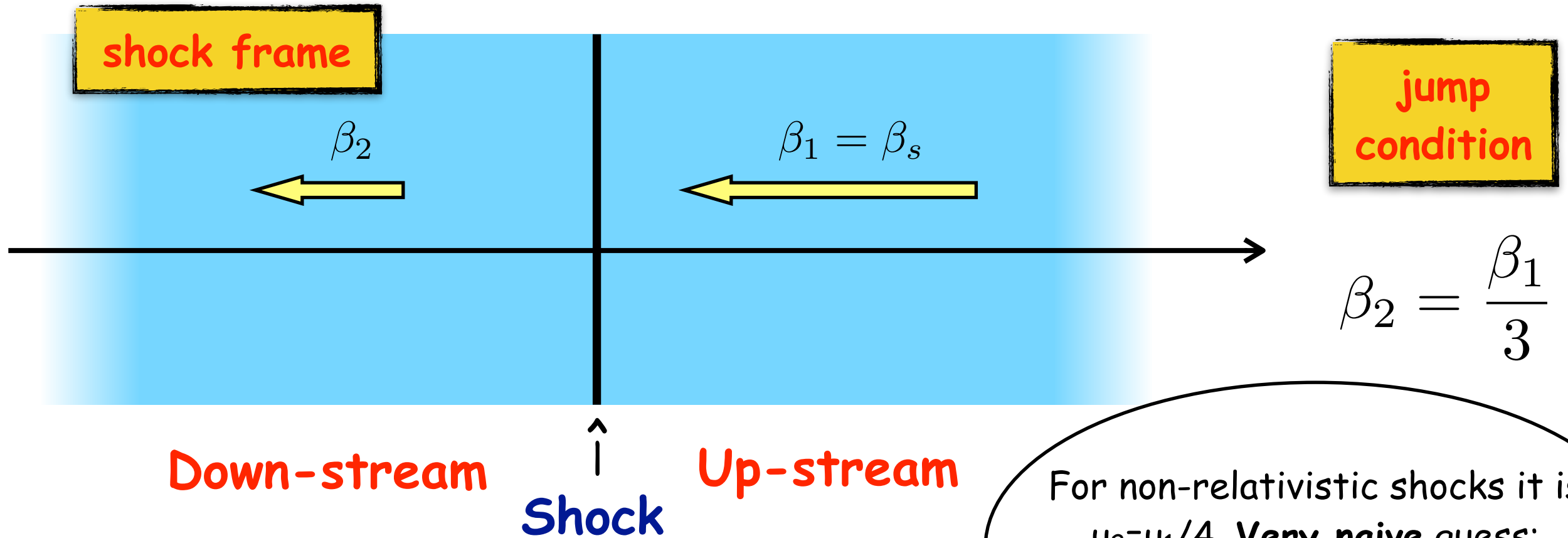
relative speed

$$\beta_{rel} = \frac{\beta_1 - \beta_2}{1 - \beta_1 \beta_2}$$

For non-relativistic shocks it is $u_2 = u_1/4$. Very naive guess: spectra are in this case steeper than E^{-2} ?



Ultra-relativistic shocks



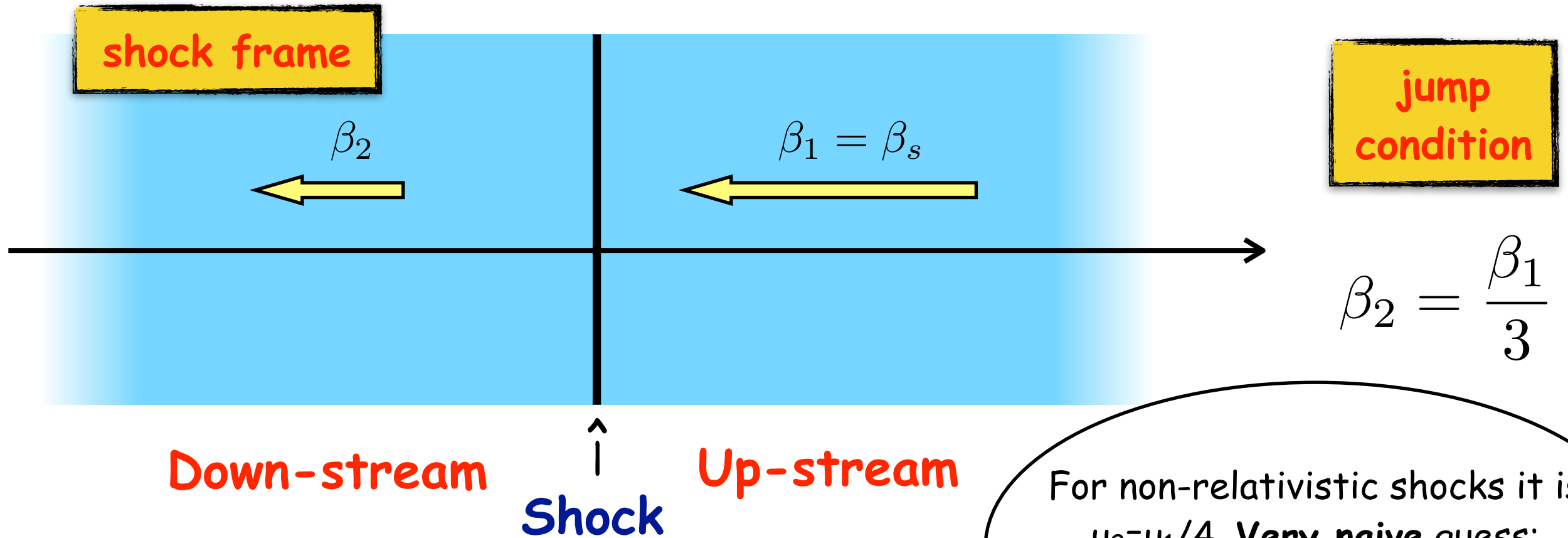
relative speed

$$\beta_{rel} = \frac{\beta_1 - \beta_2}{1 - \beta_1 \beta_2} = \frac{\frac{2}{3} \beta_1}{1 - \frac{\beta_1^2}{3}}$$

For non-relativistic shocks it is $u_2 = u_1/4$. Very naive guess: spectra are in this case steeper than E^{-2} ?



Ultra-relativistic shocks



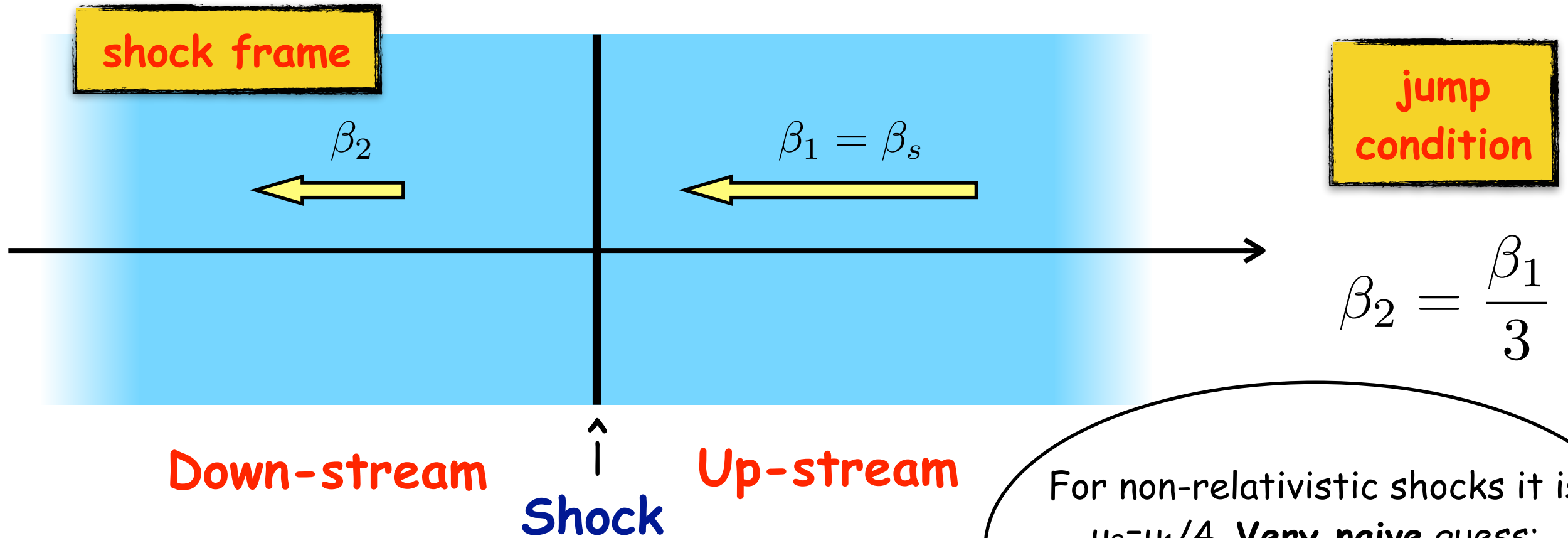
relative speed

$$\beta_{rel} = \frac{\beta_1 - \beta_2}{1 - \beta_1 \beta_2} = \frac{\frac{2}{3} \beta_1}{1 - \frac{\beta_1^2}{3}} = \frac{\left(1 - \frac{1}{\Gamma_s^2}\right)^{1/2}}{\left(1 + \frac{1}{2\Gamma_s^2}\right)}$$

For non-relativistic shocks it is $u_2 = u_1/4$. Very naive guess: spectra are in this case steeper than E^{-2} ?



Ultra-relativistic shocks



relative speed

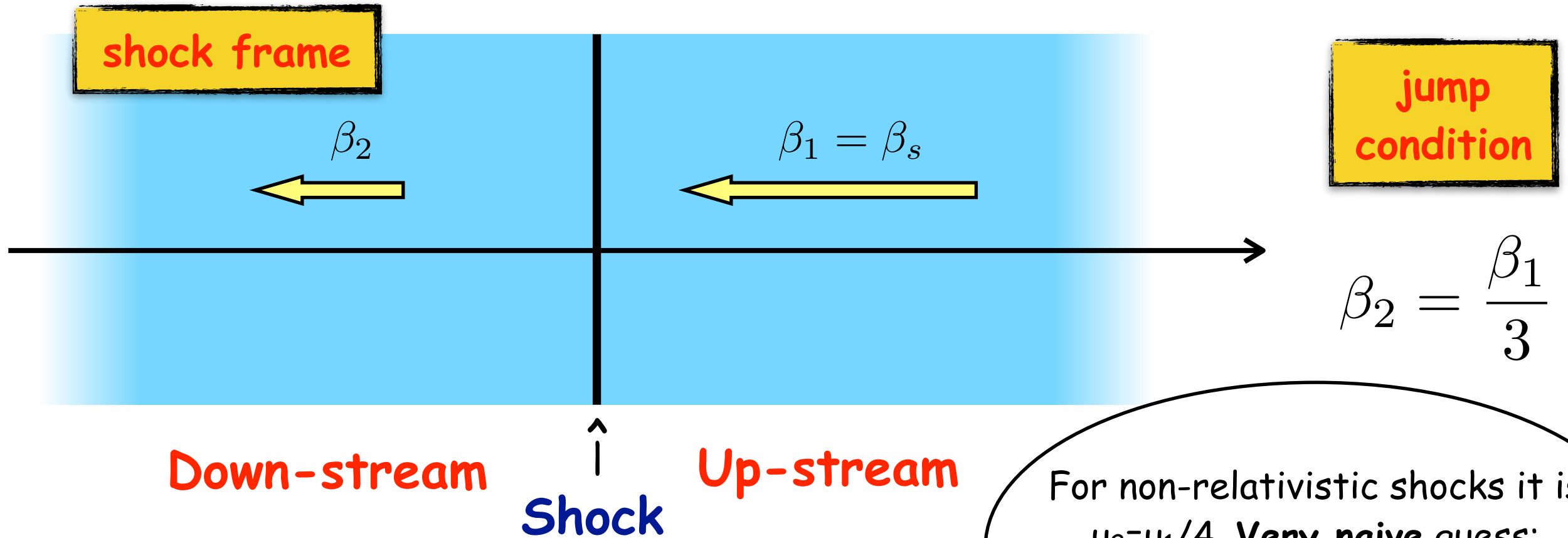
$$\beta_{rel} = \frac{\beta_1 - \beta_2}{1 - \beta_1 \beta_2} = \frac{\frac{2}{3} \beta_1}{1 - \frac{\beta_1^2}{3}} =$$

$$= \frac{\left(1 - \frac{1}{\Gamma_s^2}\right)^{1/2}}{\left(1 + \frac{1}{2\Gamma_s^2}\right)} \approx \left(1 - \frac{1}{2\Gamma_s^2}\right) \left(1 - \frac{1}{2\Gamma_s^2}\right) \approx 1 - \frac{1}{\Gamma_s^2}$$

For non-relativistic shocks it is $u_2 = u_1/4$. **Very naive** guess: spectra are in this case steeper than E^{-2} ?



Ultra-relativistic shocks



relative speed

$$\beta_{rel} = \frac{\beta_1 - \beta_2}{1 - \beta_1 \beta_2} = \frac{\frac{2}{3} \beta_1}{1 - \frac{\beta_1^2}{3}} =$$

$$= \frac{\left(1 - \frac{1}{\Gamma_s^2}\right)^{1/2}}{\left(1 + \frac{1}{2\Gamma_s^2}\right)} \approx \left(1 - \frac{1}{2\Gamma_s^2}\right) \left(1 - \frac{1}{2\Gamma_s^2}\right) \approx 1 - \frac{1}{\Gamma_s^2}$$

For non-relativistic shocks it is $u_2 = u_1/4$. Very naive guess: spectra are in this case steeper than E^{-2} ?



Ultra-relativistic shocks: summary

shock velocity

$$\beta_s \approx 1 - \frac{1}{2\Gamma_s^2}$$

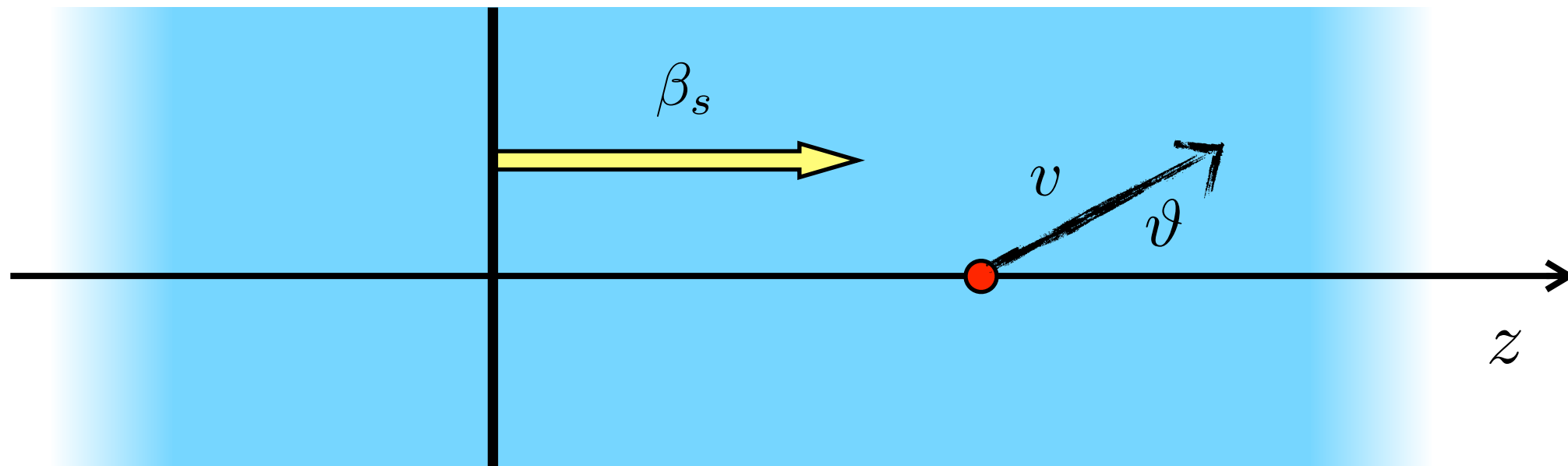
relative speed

$$\beta_{rel} \approx 1 - \frac{1}{\Gamma_s^2}$$

jump condition

$$\beta_2 = \frac{\beta_1}{3}$$

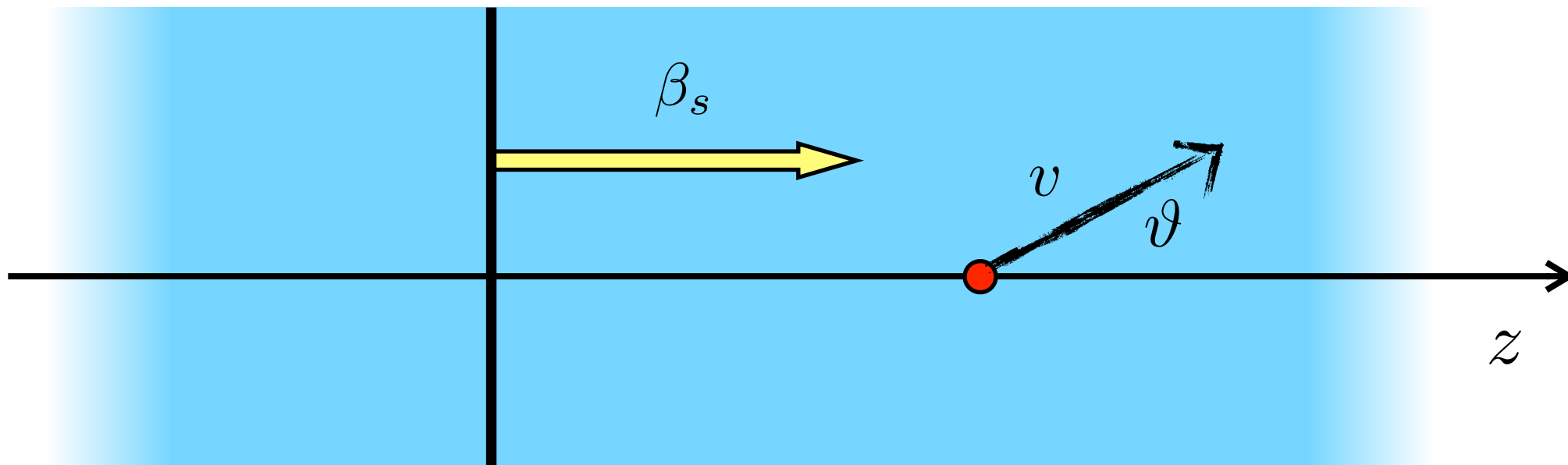
Loss cone



Down-stream $\uparrow$
Shock

Up-stream

Loss cone



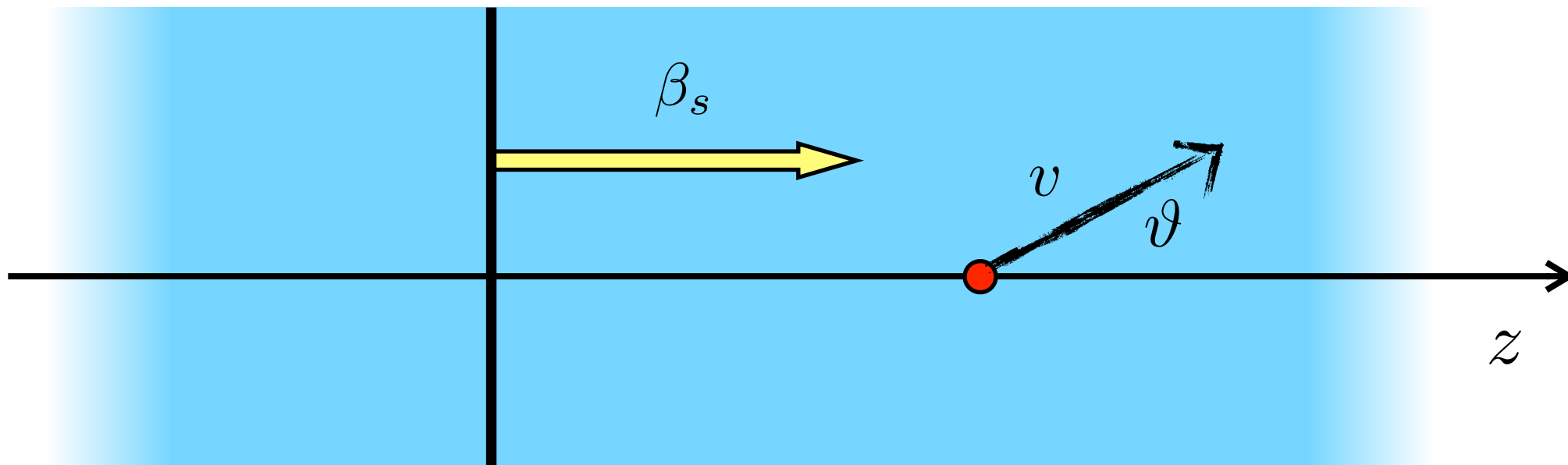
Down-stream $\uparrow$
Shock

Up-stream

will the shock catch the particle?

$$\beta_s > v_z \sim \cos \vartheta$$

Loss cone



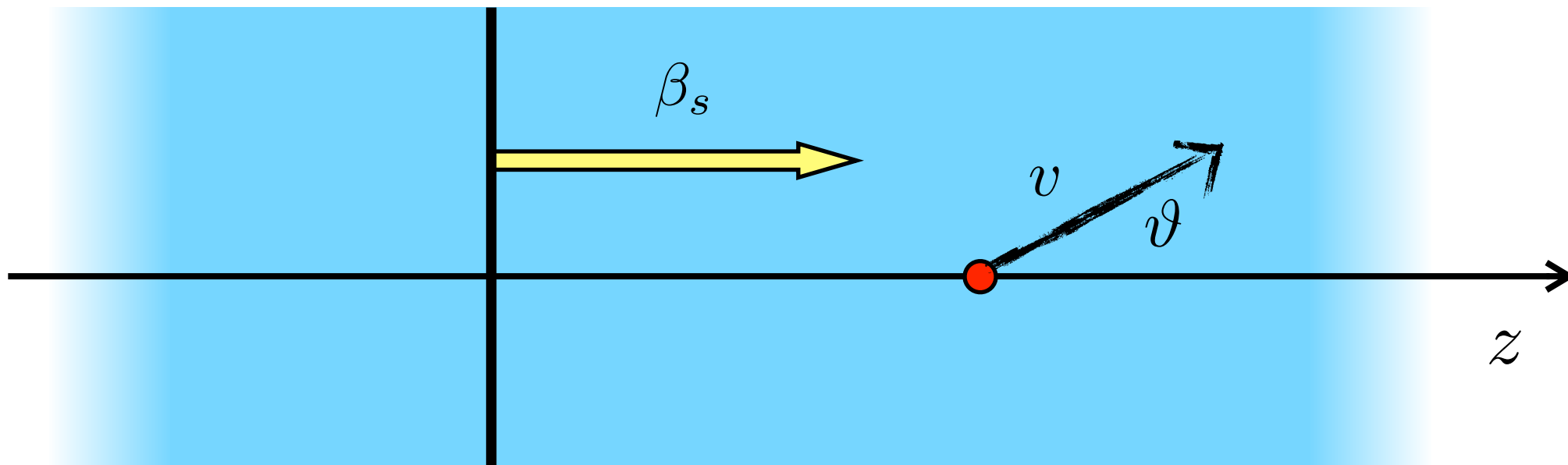
Down-stream $\uparrow$
Shock

Up-stream

will the shock catch the particle?

$$1 - \frac{1}{2\Gamma_s^2} \sim \beta_s > v_z \sim \cos \vartheta$$

Loss cone



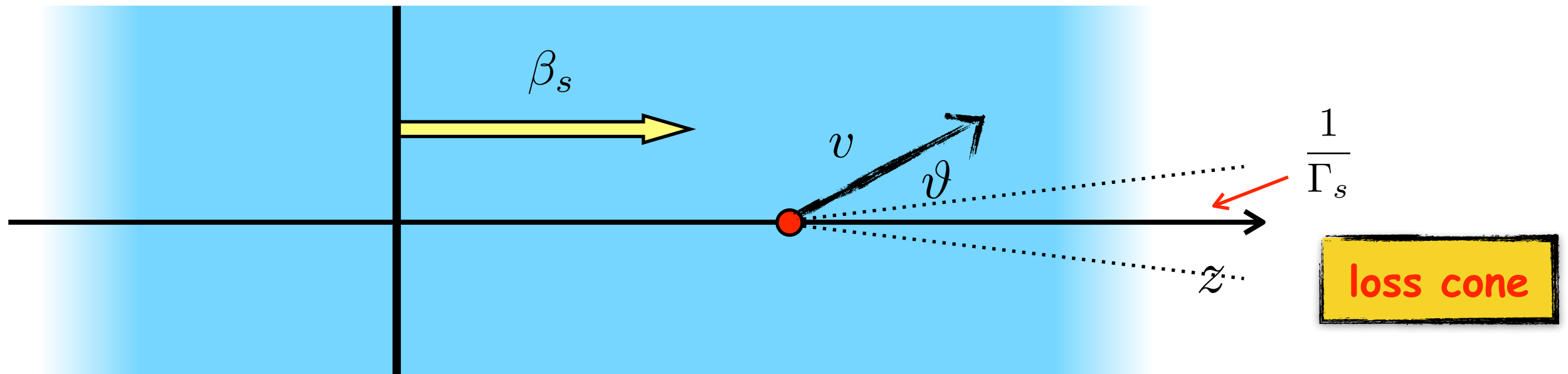
Down-stream $\uparrow$
Shock

Up-stream

will the shock catch the particle?

$$1 - \frac{1}{2\Gamma_s^2} \sim \beta_s > v_z \sim \cos \vartheta \longrightarrow \frac{1}{2\Gamma_s^2} < 1 - \cos \vartheta \sim \frac{\vartheta^2}{2}$$

Loss cone



Down-stream $\uparrow$
Shock

Up-stream

will the shock catch the particle?

$$1 - \frac{1}{2\Gamma_s^2} \sim \beta_s > v_z \sim \cos \vartheta \longrightarrow \frac{1}{2\Gamma_s^2} < 1 - \cos \vartheta \sim \frac{\vartheta^2}{2}$$

$$\vartheta > \frac{1}{\Gamma_s}$$

Anisotropy

Consequences:

Anisotropy

Consequences:

Anisotropy

Consequences:

- particles that just crossed the shock (down→up) are within the loss cone ($\sim 1/\Gamma_s$)

Anisotropy

Consequences:

- particles that just crossed the shock (down→up) are within the loss cone ($\sim 1/\Gamma_s$)
- particles deflected out of the loss cone are very quickly caught by the shock

Anisotropy

Consequences:

- particles that just crossed the shock (down→up) are within the loss cone ($\sim 1/\Gamma_s$)
- particles deflected out of the loss cone are very quickly caught by the shock
- it can be shown that the deflection outside of the loss cone is at most $\sim 1/\Gamma_s$

Anisotropy

Consequences:

- particles that just crossed the shock (down→up) are within the loss cone ($\sim 1/\Gamma_s$)
- particles deflected out of the loss cone are very quickly caught by the shock
- it can be shown that the deflection outside of the loss cone is at most $\sim 1/\Gamma_s$
- Upstream particles are all directed within a cone of opening angle $\sim 2/\Gamma_s$

Anisotropy

Consequences:

- particles that just crossed the shock (down→up) are within the loss cone ($\sim 1/\Gamma_s$)
- particles deflected out of the loss cone are very quickly caught by the shock
- it can be shown that the deflection outside of the loss cone is at most $\sim 1/\Gamma_s$
- Upstream particles are all directed within a cone of opening angle $\sim 2/\Gamma_s$

The particle distribution function upstream of the shock is very anisotropic
—> the transport is NOT diffusive

Anisotropy

Consequences:

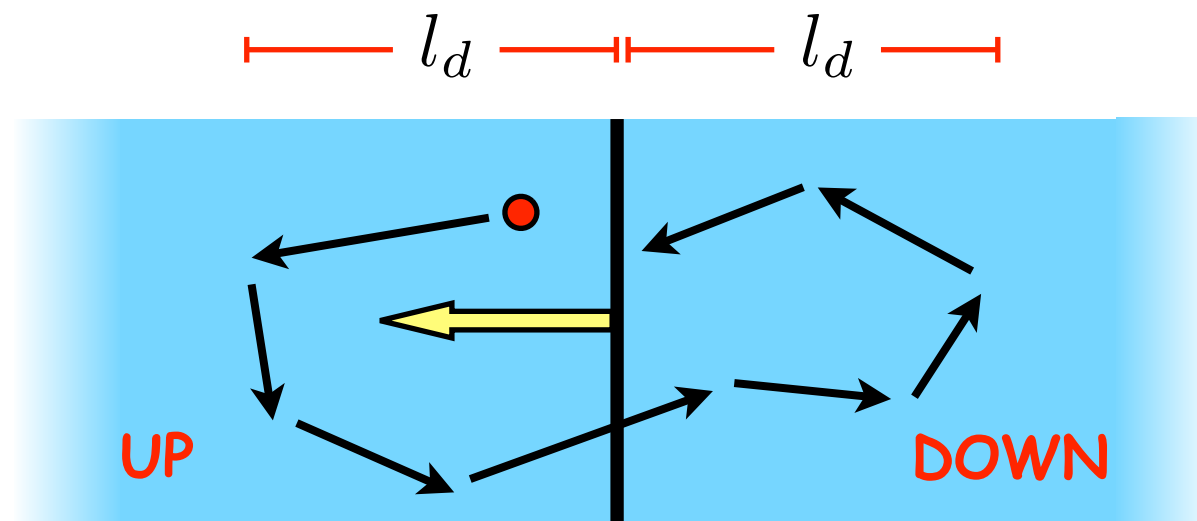
- particles that just crossed the shock (down→up) are within the loss cone ($\sim 1/\Gamma_s$)
- particles deflected out of the loss cone are very quickly caught by the shock
- it can be shown that the deflection outside of the loss cone is at most $\sim 1/\Gamma_s$
- Upstream particles are all directed within a cone of opening angle $\sim 2/\Gamma_s$

The particle distribution function upstream of the shock is very anisotropic
→ the transport is NOT diffusive

it can be demonstrate that also downstream particles are highly anisotropic

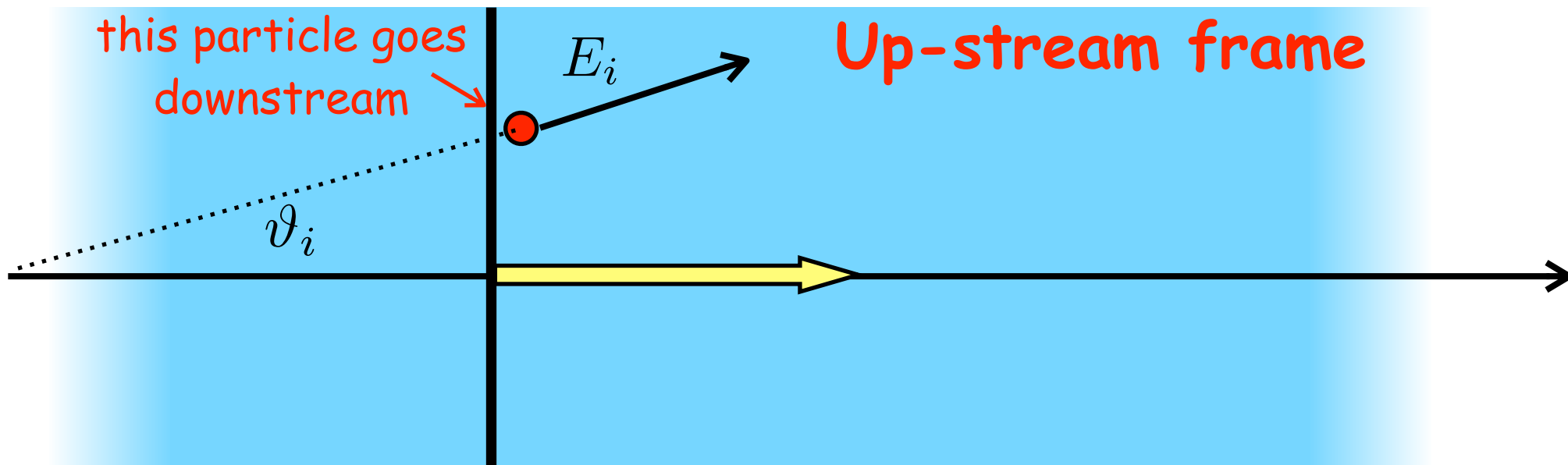
Key aspects of DSA at ~~non~~-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream

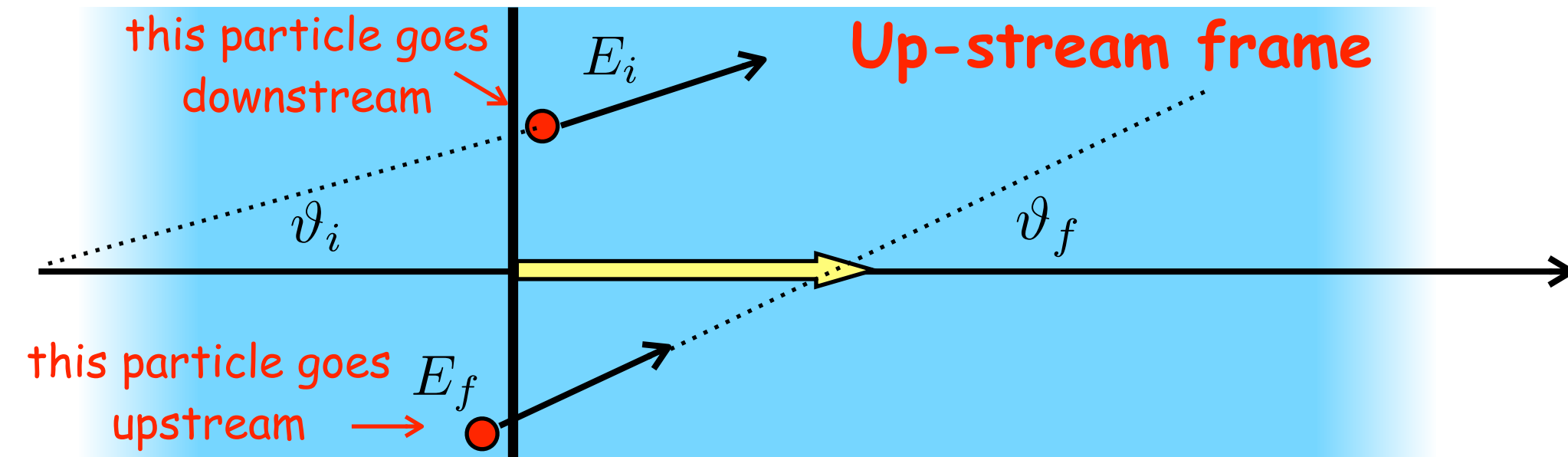


~~non-~~ diffusive transport → ~~isotropic~~

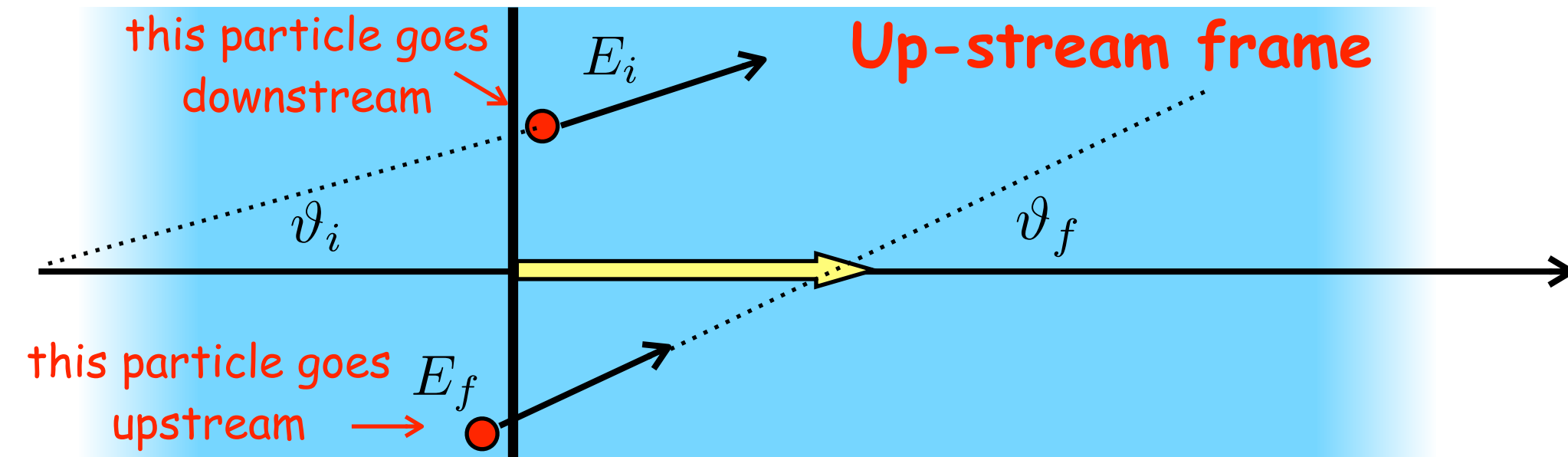
Energy gain in a cycle



Energy gain in a cycle



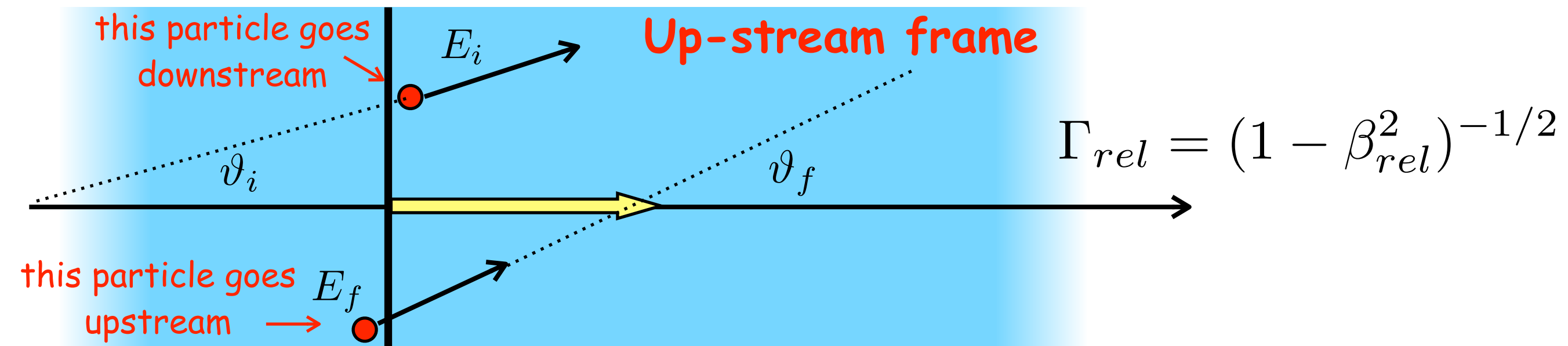
Energy gain in a cycle



primed quantities are in the downstream frame

$$E'_i = E'_f$$

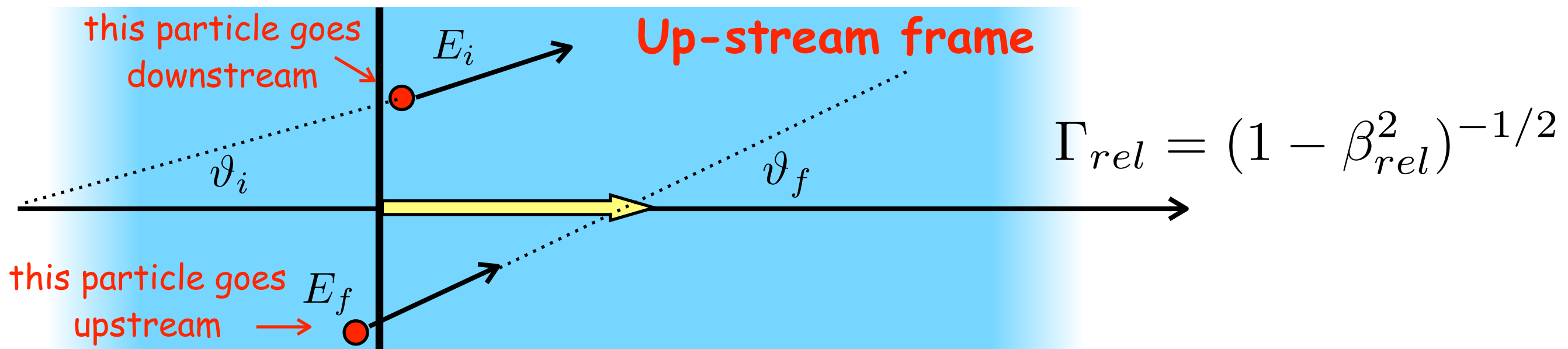
Energy gain in a cycle



primed quantities are in the downstream frame

$$\Gamma_{rel} (E_i - \beta_{rel} p_{z,i}) = E'_i = E'_f = \Gamma_{rel} (E_f - \beta_{rel} p_{z,f})$$

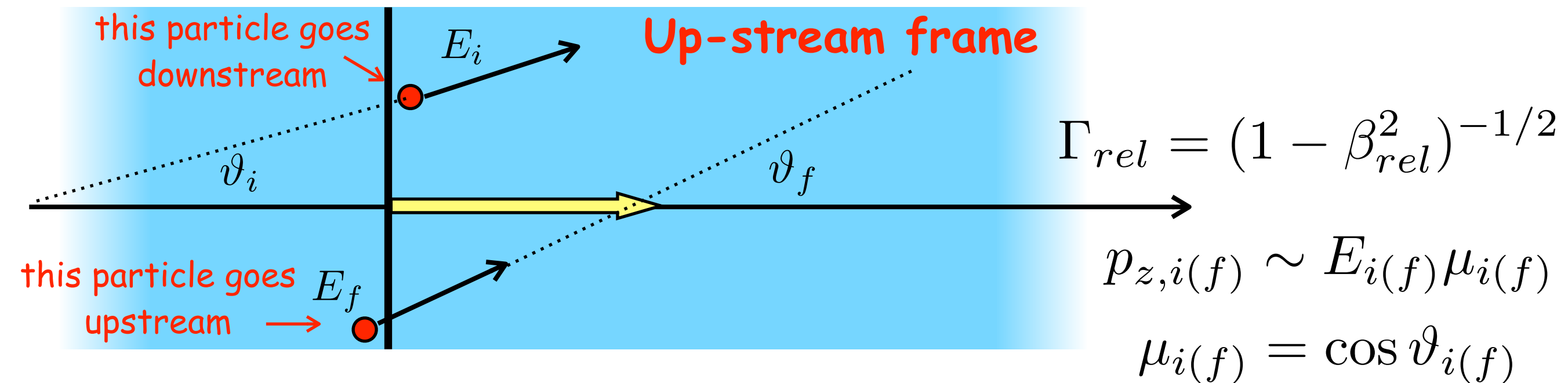
Energy gain in a cycle



primed quantities are in the downstream frame

$$\cancel{\Gamma}_{rel} (E_i - \beta_{rel} p_{z,i}) = E'_i = E'_f = \cancel{\Gamma}_{rel} (E_f - \beta_{rel} p_{z,f})$$

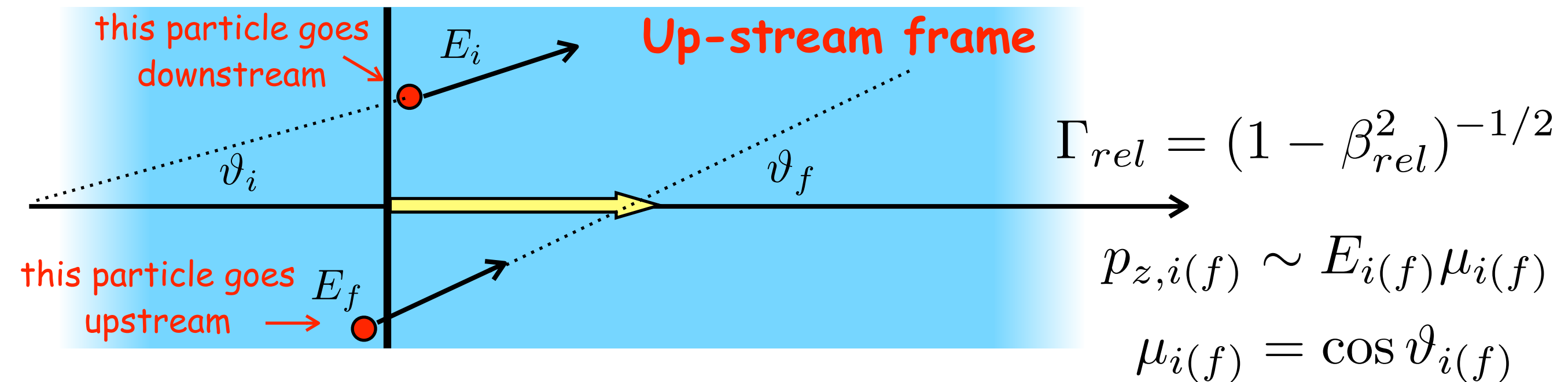
Energy gain in a cycle



primed quantities are in the downstream frame

$$\cancel{\Gamma}_{rel} (E_i - \beta_{rel} p_{z,i}) = E'_i = E'_f = \cancel{\Gamma}_{rel} (E_f - \beta_{rel} p_{z,f})$$

Energy gain in a cycle

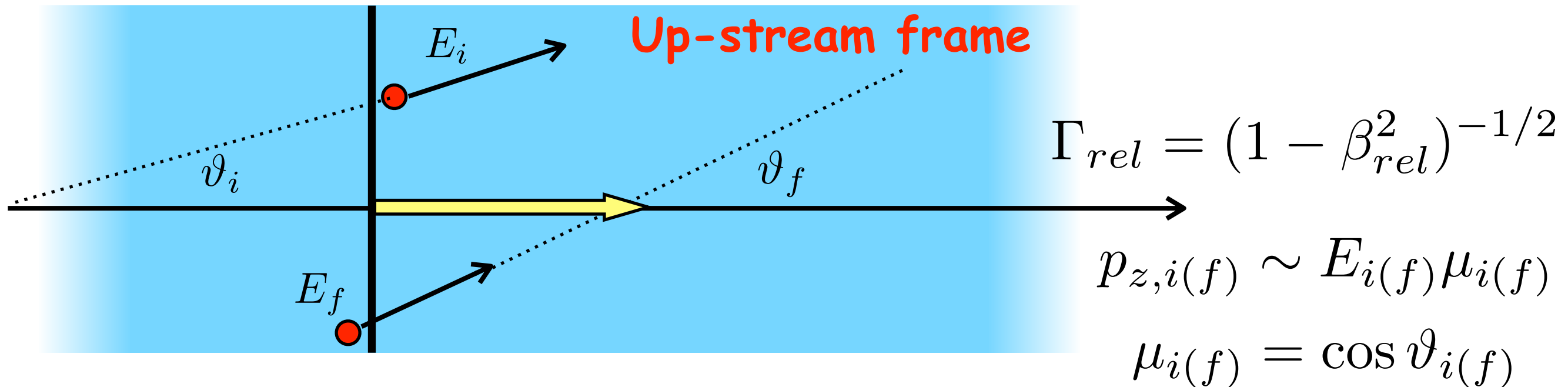


primed quantities are in the downstream frame

$$\cancel{\Gamma}_{rel} (E_i - \beta_{rel} p_{z,i}) = E'_i = E'_f = \cancel{\Gamma}_{rel} (E_f - \beta_{rel} p_{z,f})$$

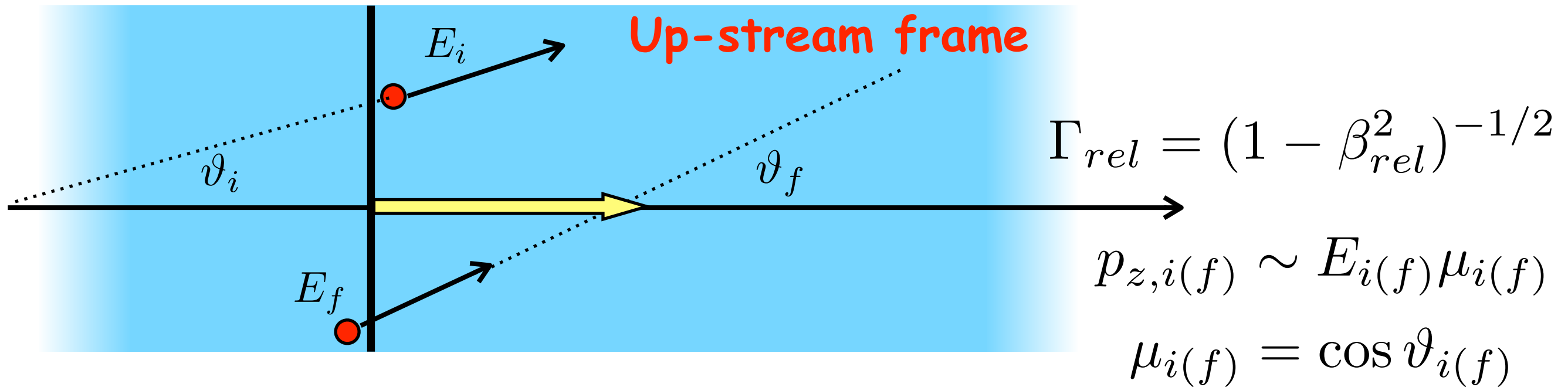
$$\frac{E_f}{E_i} \sim \frac{1 - \beta_{rel} \mu_i}{1 - \beta_{rel} \mu_f}$$

Energy gain in a cycle



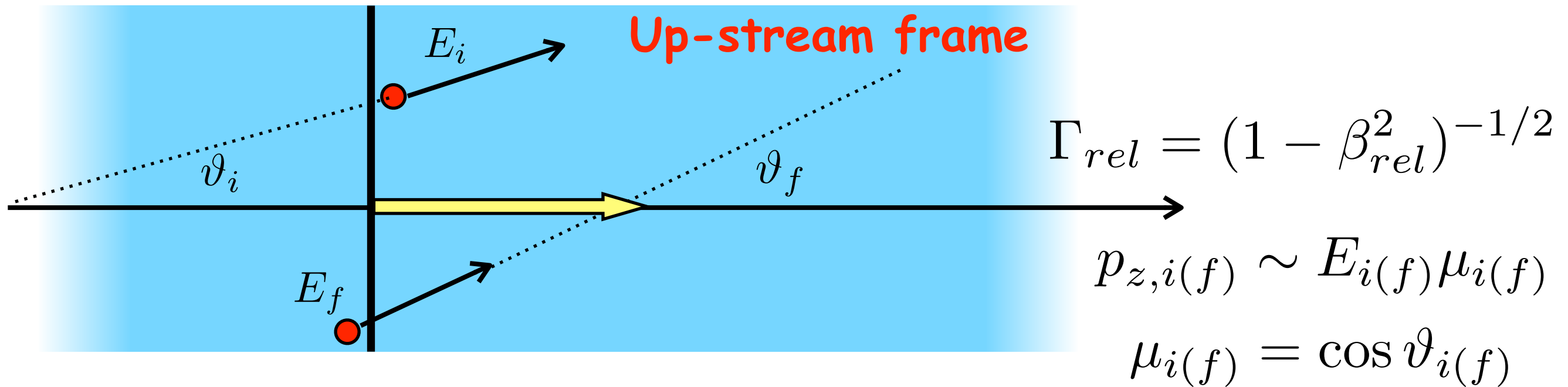
$$\frac{E_f}{E_i} \sim \frac{1 - \beta_{rel} \mu_i}{1 - \beta_{rel} \mu_f}$$

Energy gain in a cycle



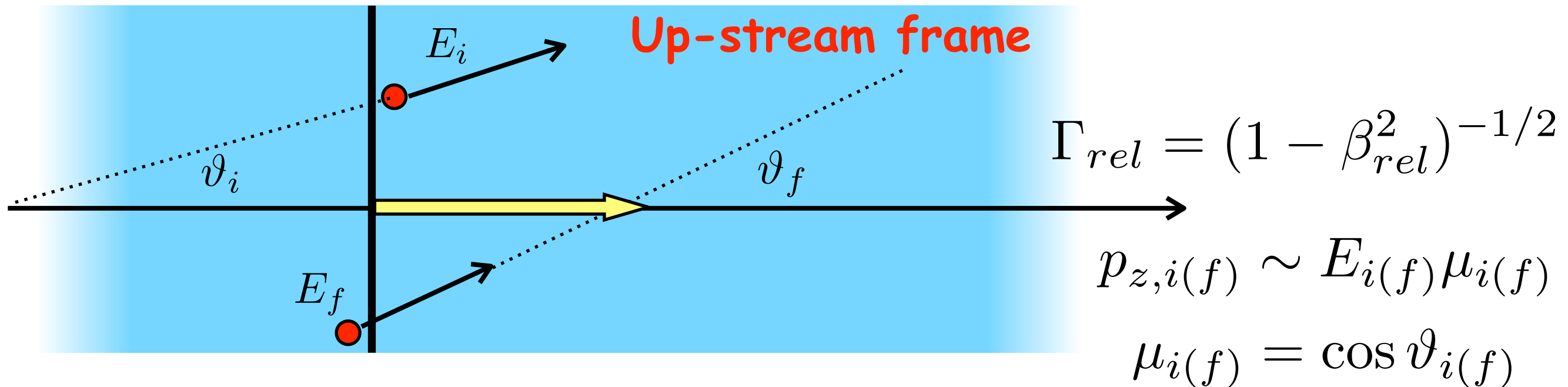
$$\frac{E_f}{E_i} \sim \frac{1 - \beta_{rel} \mu_i}{1 - \beta_{rel} \mu_f} = \frac{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_i^2}{2}\right)}{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_f^2}{2}\right)}$$

Energy gain in a cycle



$$\frac{E_f}{E_i} \sim \frac{1 - \beta_{rel} \mu_i}{1 - \beta_{rel} \mu_f} = \frac{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_i^2}{2}\right)}{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_f^2}{2}\right)} = \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

Energy gain in a cycle

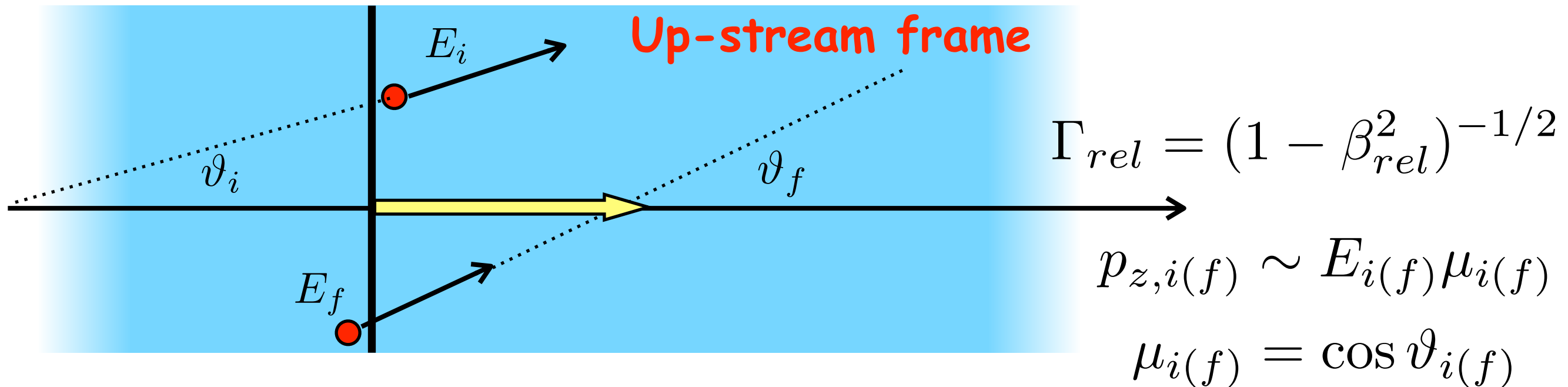


$$\frac{E_f}{E_i} \sim \frac{1 - \beta_{rel} \mu_i}{1 - \beta_{rel} \mu_f} = \frac{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_i^2}{2}\right)}{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_f^2}{2}\right)} = \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

up $\rightarrow$ down

$$\frac{1}{\Gamma_s} < \vartheta_i < \frac{2}{\Gamma_s} \longrightarrow 1 < \vartheta_i \Gamma_s < 2$$

Energy gain in a cycle



$$\frac{E_f}{E_i} \sim \frac{1 - \beta_{rel} \mu_i}{1 - \beta_{rel} \mu_f} = \frac{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_i^2}{2}\right)}{1 - \left(1 - \frac{1}{\Gamma_s^2}\right) \left(1 + \frac{\vartheta_f^2}{2}\right)} = \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

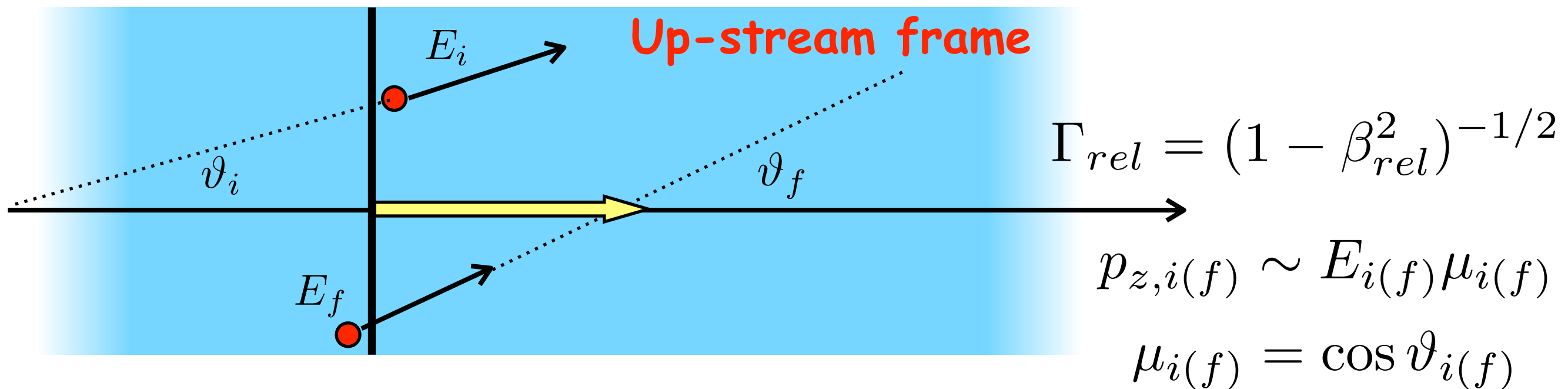
up $\rightarrow$ down

$$\frac{1}{\Gamma_s} < \vartheta_i < \frac{2}{\Gamma_s} \longrightarrow 1 < \vartheta_i \Gamma_s < 2$$

down $\rightarrow$ up

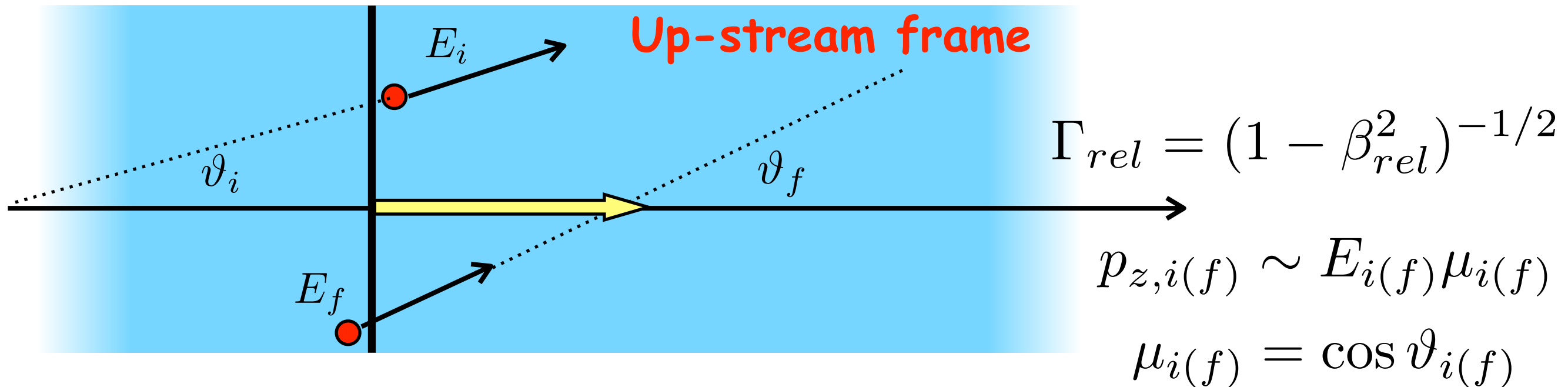
$$0 < \vartheta_f < \frac{1}{\Gamma_s} \longrightarrow 0 < \vartheta_f \Gamma_s < 1$$

Energy gain in a cycle



$$\left. \begin{aligned} \frac{E_f}{E_i} &\sim \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2} \\ 1 &< \vartheta_i \Gamma_s < 2 \\ 0 &< \vartheta_f \Gamma_s < 1 \end{aligned} \right\}$$

Energy gain in a cycle



$$\frac{E_f}{E_i} \sim \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

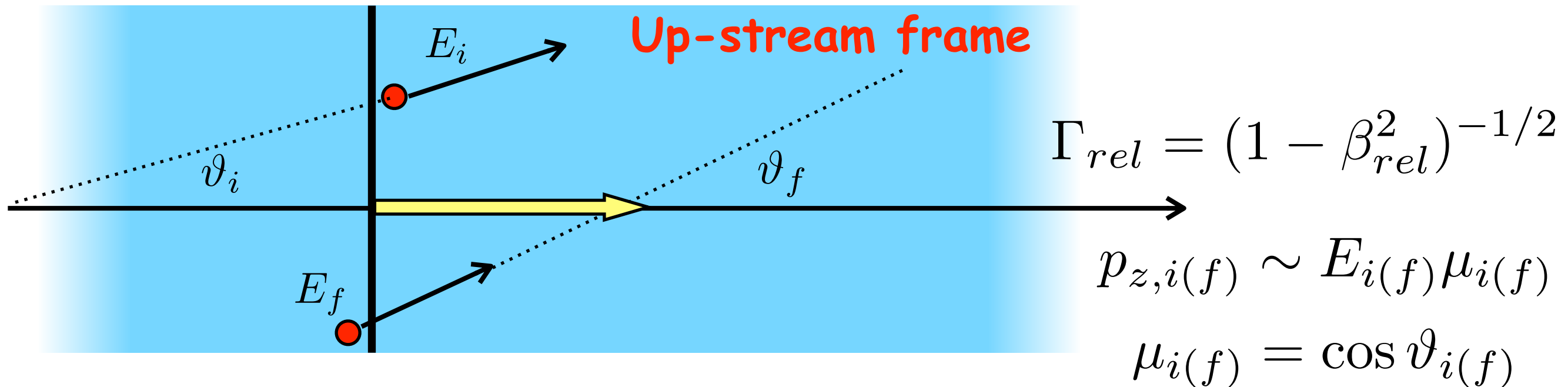
$$1 < \vartheta_i \Gamma_s < 2$$

$$0 < \vartheta_f \Gamma_s < 1$$

naively...

$$1 < \frac{E_f}{E_i} < 3$$

Energy gain in a cycle



$$\frac{E_f}{E_i} \sim \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

naively...

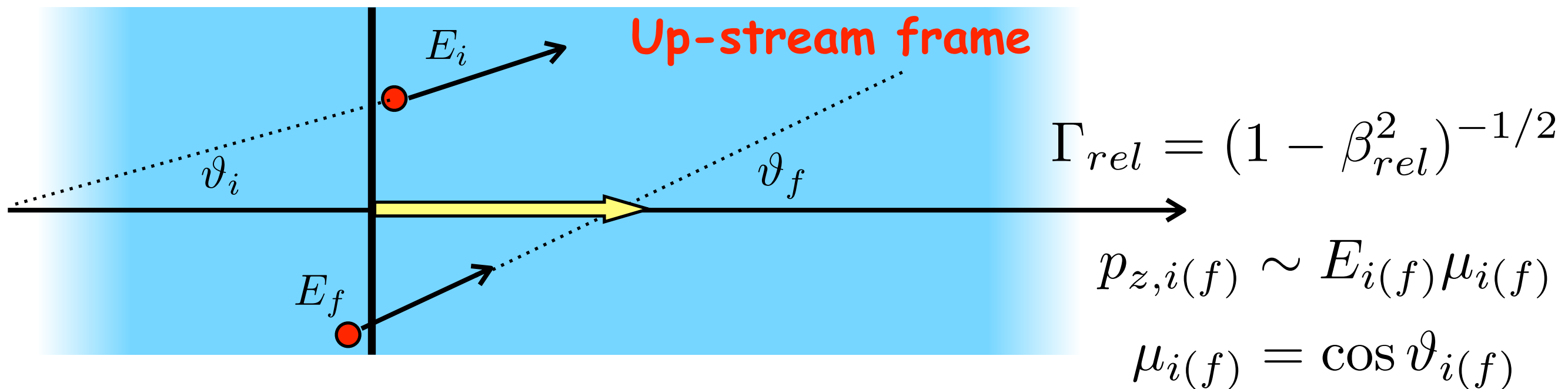
$$1 < \vartheta_i \Gamma_s < 2$$

$$1 < \frac{E_f}{E_i} < 3$$

$$0 < \vartheta_f \Gamma_s < 1$$

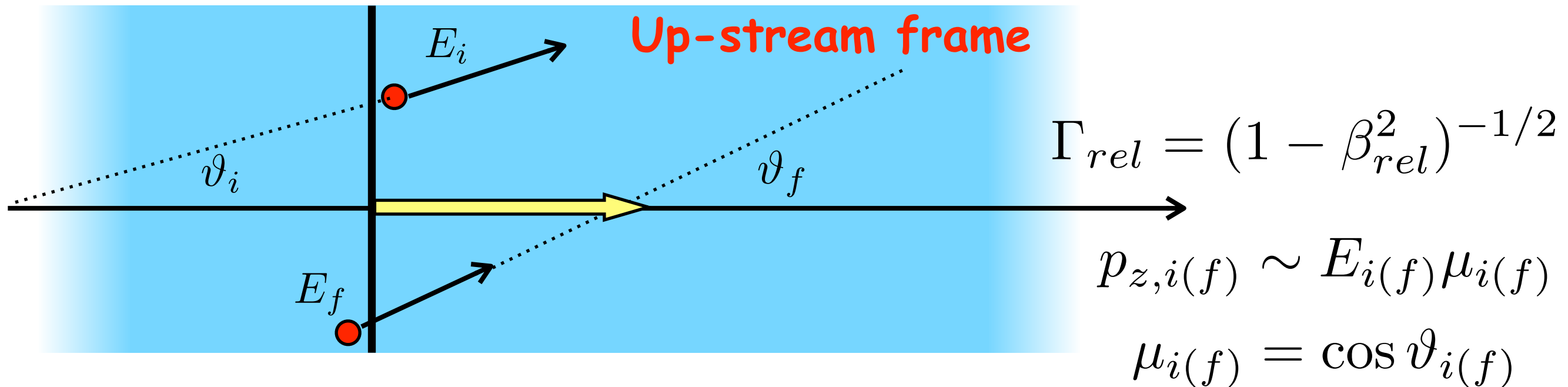
as for non-relativistic shocks, particles always gain energy, but now the gain can be large!

Energy gain in a cycle



$$\left. \begin{aligned} \frac{E_f}{E_i} &\sim \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2} \\ 1 &< \vartheta_i \Gamma_s < 2 \\ 0 &< \vartheta_f \Gamma_s < 1 \end{aligned} \right\}$$

Energy gain in a cycle



$$\frac{E_f}{E_i} \sim \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

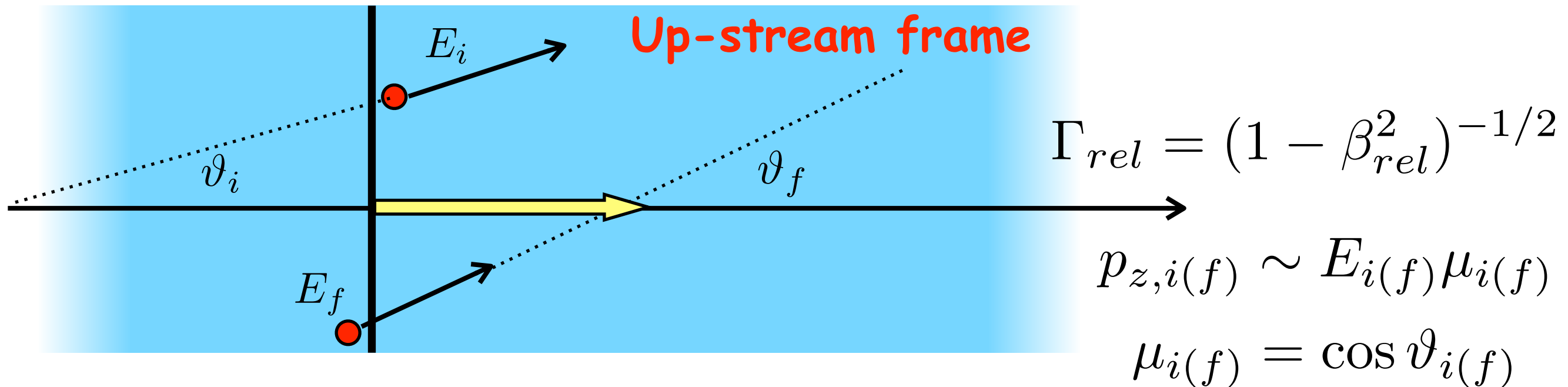
$$1 < \vartheta_i \Gamma_s < 2$$

$$0 < \vartheta_f \Gamma_s < 1$$

it can be shown that:

$$\frac{E_f}{E_i} \approx 2 \rightarrow \frac{\Delta E}{E} \approx 1$$

Energy gain in a cycle



$$\frac{E_f}{E_i} \sim \frac{2 + \Gamma_s^2 \vartheta_i^2}{2 + \Gamma_s^2 \vartheta_f^2}$$

$$1 < \vartheta_i \Gamma_s < 2$$

$$0 < \vartheta_f \Gamma_s < 1$$

it can be shown that:

$$\frac{E_f}{E_i} \approx 2 \rightarrow \frac{\Delta E}{E} \approx 1$$

the energy gain is large!

First shock crossing: the initial boost

So far, we always considered particles entering the upstream region crossing the shock from the downstream one $\rightarrow$ particle velocity upstream is very close to the shock normal (loss cone)

First shock crossing: the initial boost

So far, we always considered particles entering the upstream region crossing the shock from the downstream one → particle velocity upstream is very close to the shock normal (loss cone)

first interaction with the shock

→ the particle can move at any angle with respect to the shock normal!

First shock crossing: the initial boost

So far, we always considered particles entering the upstream region crossing the shock from the downstream one → particle velocity upstream is very close to the shock normal (loss cone)

first interaction with the shock

→ the particle can move at any angle with respect to the shock normal!

$$\frac{E_f}{E_i} = \Gamma_{rel}^2 (1 - \beta_{rel} \mu_i) (1 + \beta_{rel} \mu'_f)$$

First shock crossing: the initial boost

So far, we always considered particles entering the upstream region crossing the shock from the downstream one → particle velocity upstream is very close to the shock normal (loss cone)

first interaction with the shock

→ the particle can move at any angle with respect to the shock normal!

$$\frac{E_f}{E_i} = \Gamma_{rel}^2 (1 - \beta_{rel} \mu_i) (1 + \beta_{rel} \mu'_f) \approx \Gamma_{rel}^2 \sim \Gamma_s^2$$

First shock crossing: the initial boost

So far, we always considered particles entering the upstream region crossing the shock from the downstream one → particle velocity upstream is very close to the shock normal (loss cone)

first interaction with the shock

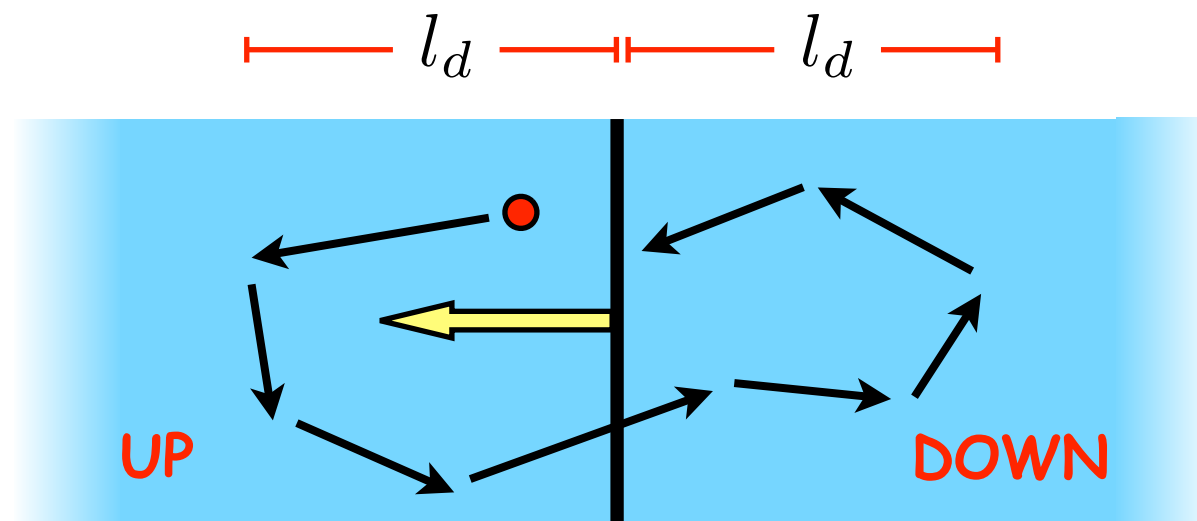
→ the particle can move at any angle with respect to the shock normal!

$$\frac{E_f}{E_i} = \Gamma_{rel}^2 (1 - \beta_{rel} \mu_i) (1 + \beta_{rel} \mu'_f) \approx \Gamma_{rel}^2 \sim \Gamma_s^2$$

very large boost of the energy at the first crossing!

Key aspects of DSA at ~~non~~-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream



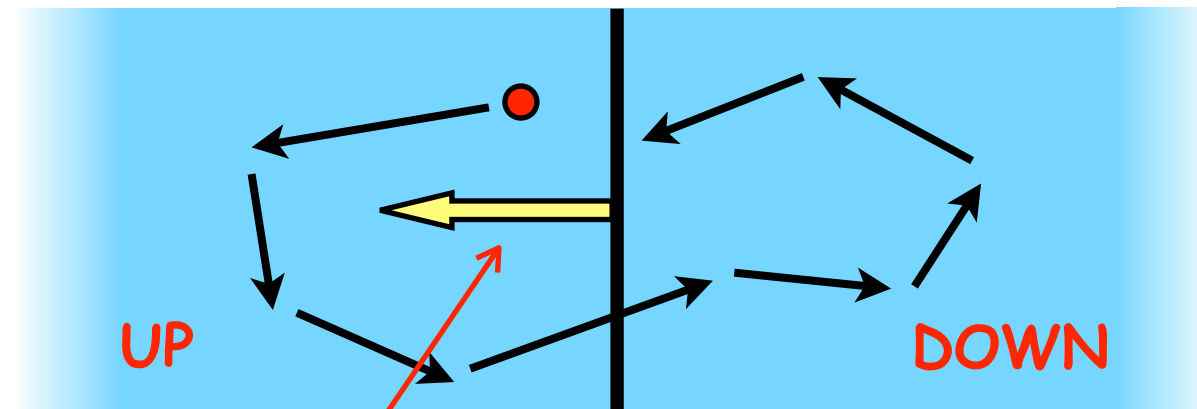
~~non-~~ diffusive transport → ~~isotropic~~

energy gain per cycle → ~~small~~ large

$$\Delta_{acc} = \Delta E / E \sim \Gamma_s^2 \quad \text{1st crossing}$$

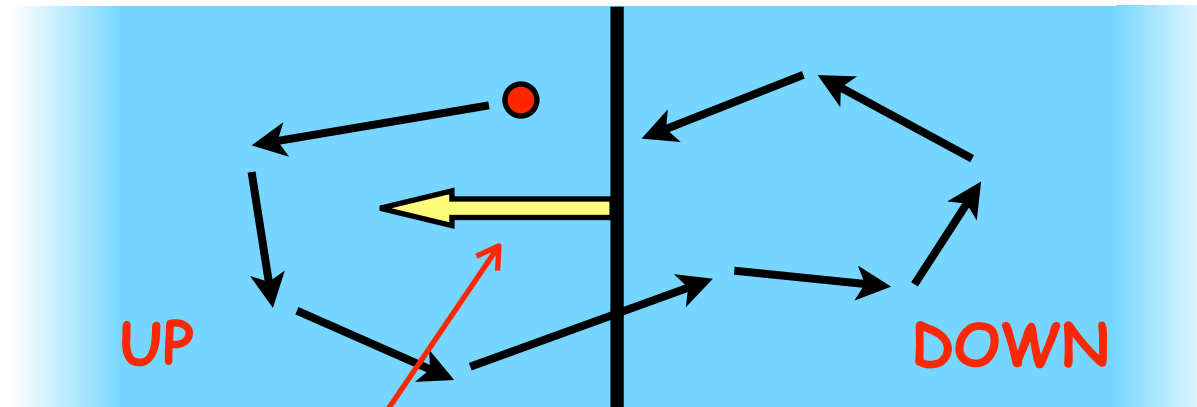
$$\Delta_{acc} = \Delta E / E \sim 1 \quad \text{all others}$$

Escape probability per cycle



$$\beta'_s = \beta_2 = \frac{\beta_s}{3}$$

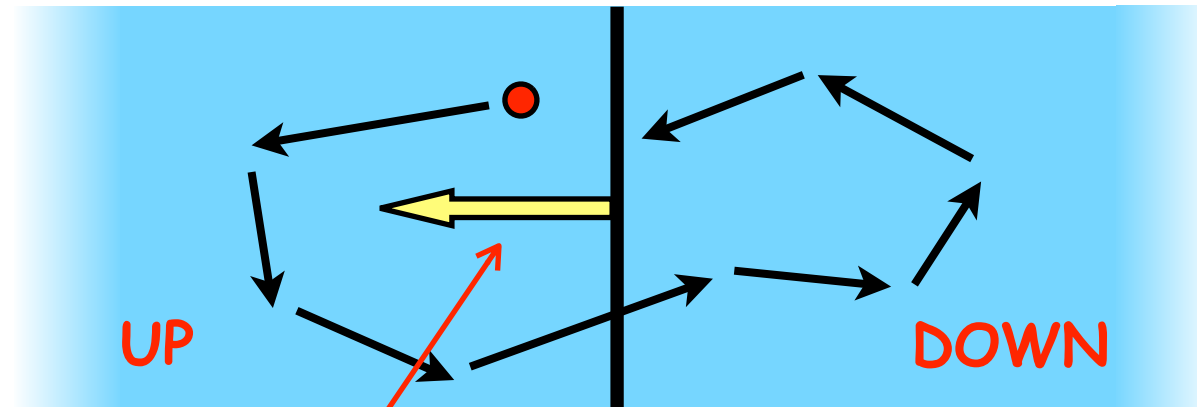
Escape probability per cycle



$$\beta'_s = \beta_2 = \frac{\beta_s}{3}$$

the particle downstream has to invert its motion (strong scattering required) and has to catch the shock that moves away at $c/3$

Escape probability per cycle



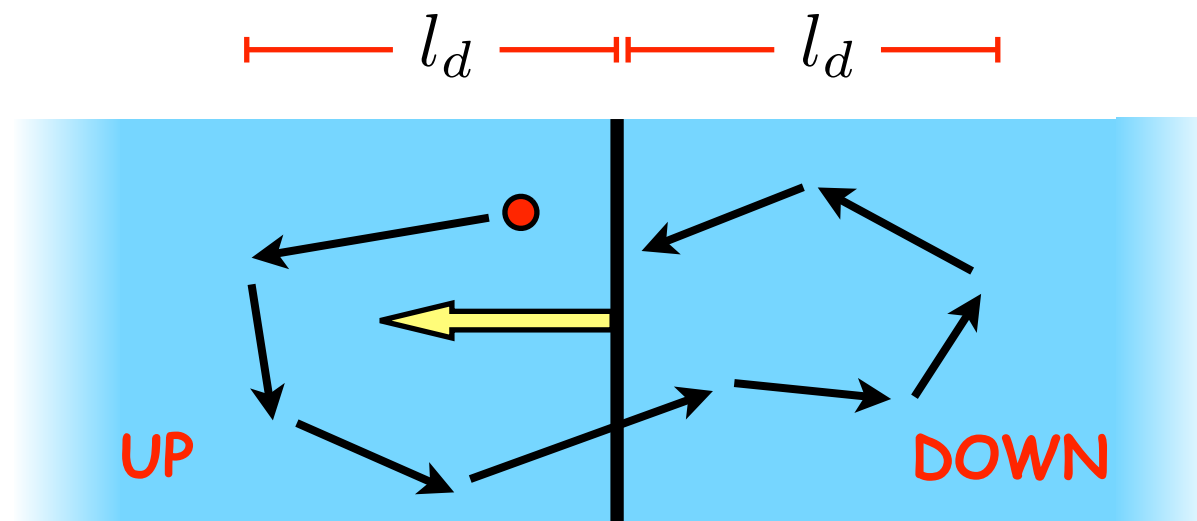
$$\beta'_s = \beta_2 = \frac{\beta_s}{3}$$

the particle downstream has to invert its motion (strong scattering required) and has to catch the shock that moves away at $c/3$

the escape probability per cycle is large!

Key aspects of DSA at ~~non~~-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream



~~non-~~ diffusive transport → ~~isotropic~~

energy gain per cycle → ~~small~~ large

escape probability per cycle → ~~small~~ large

$$\Delta_{acc} = \Delta E / E \sim \Gamma_s^2 \quad \text{1st crossing}$$

$$\Delta_{acc} = \Delta E / E \sim 1 \quad \text{all others}$$

$$P_{esc} \approx 0.5$$

Spectrum of accelerated particles

non-relativistic shocks —>

small
small

Spectrum of accelerated particles

non-relativistic shocks →

$$s = 1 + \frac{P_{esc}}{\Delta_{acc}}$$

small

small

Spectrum of accelerated particles

non-relativistic shocks →

$$s = 1 + \frac{P_{esc}}{\Delta_{acc}}$$

small

small

general expression →

$$s = 1 - \frac{\ln P_{ret}}{\ln \left(\frac{E_f}{E_i} \right)}$$

Spectrum of accelerated particles

non-relativistic shocks →

$$s = 1 + \frac{P_{esc}}{\Delta_{acc}}$$

small (pointing to P_{esc})
small (pointing to Δ_{acc})

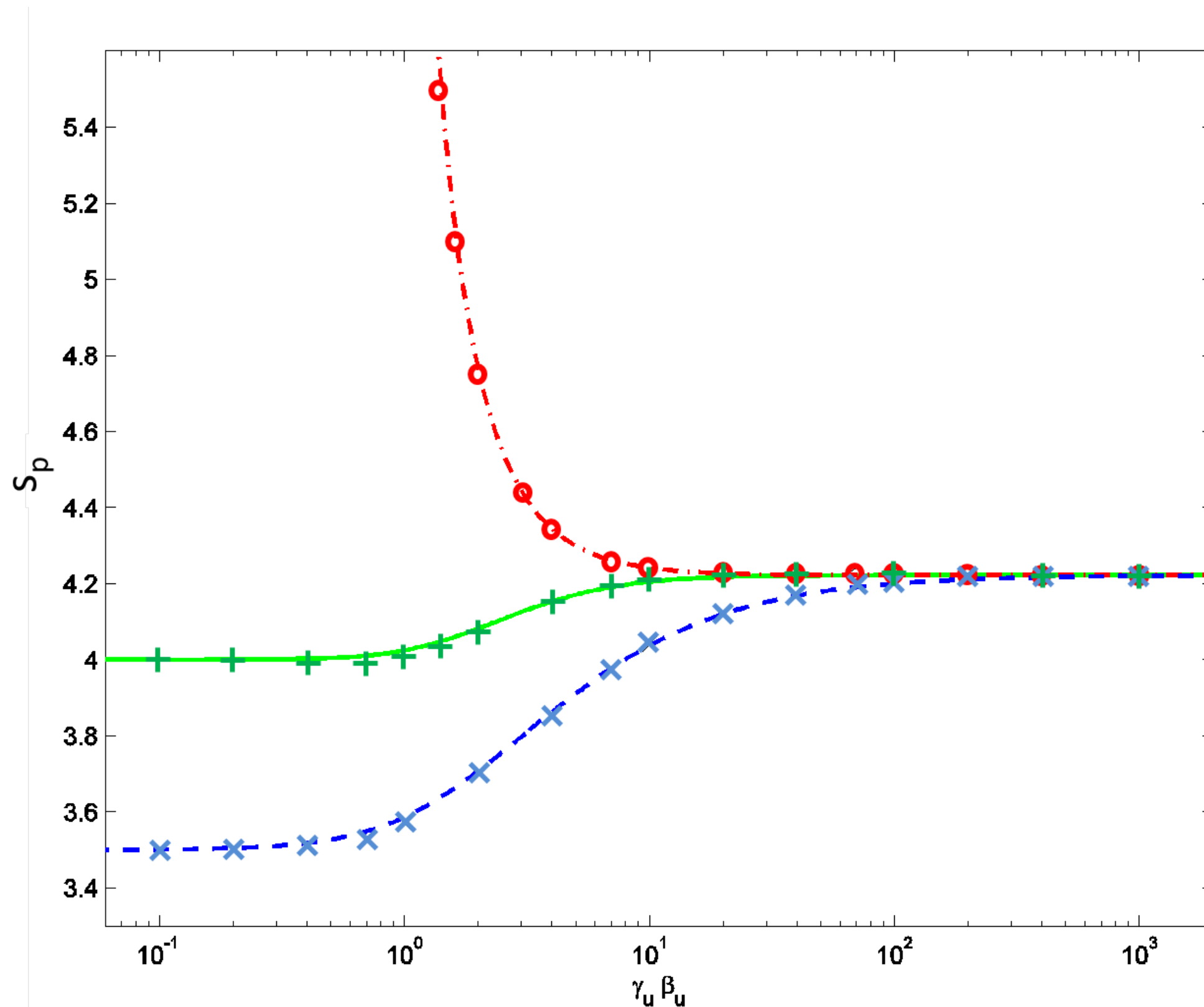
general expression →

$$s = 1 - \frac{\ln P_{ret}}{\ln \left(\frac{E_f}{E_i} \right)}$$

very roughly: s is of the order of 2

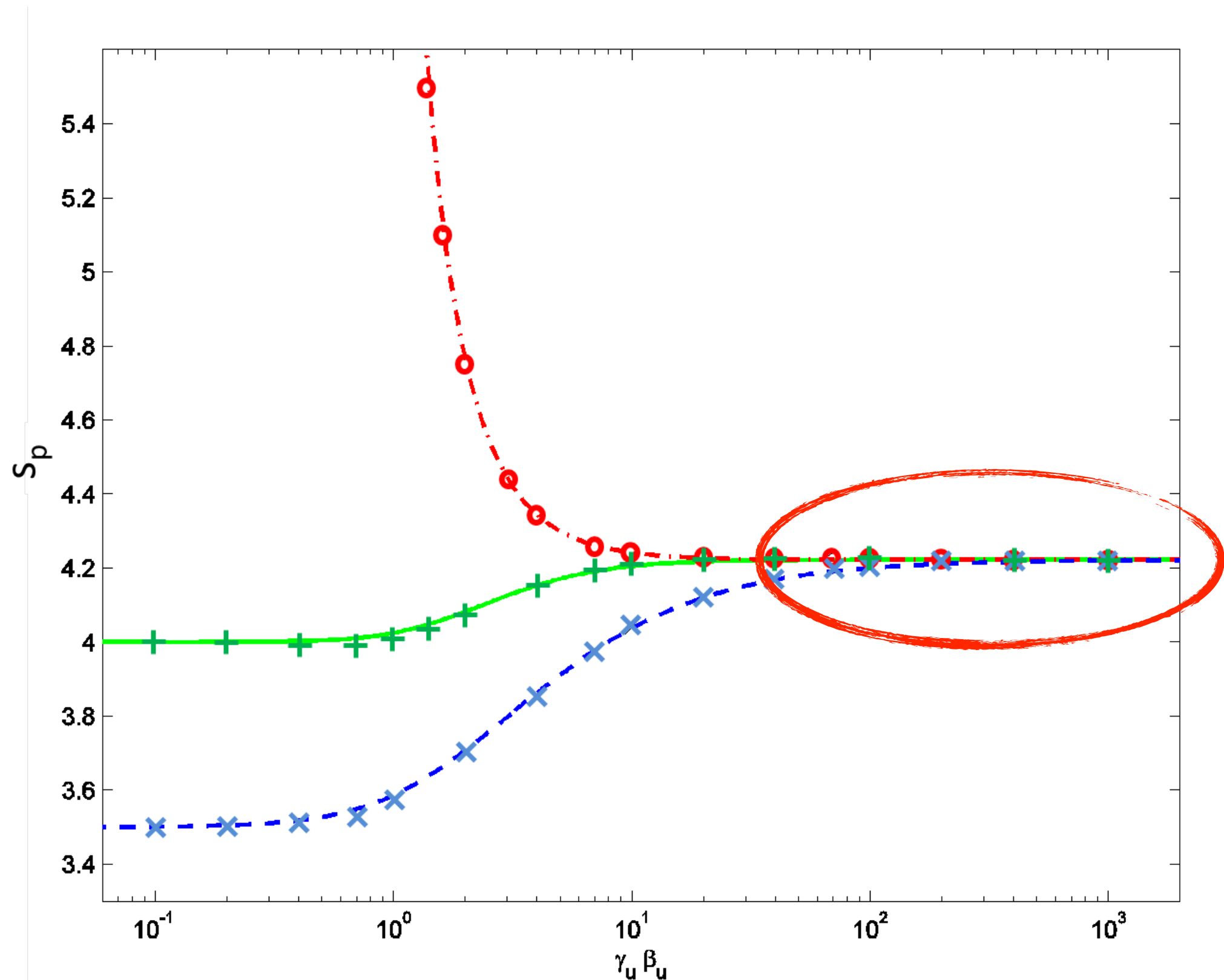
more sophisticated approaches are needed to give a more accurate answer

Spectrum of accelerated particles

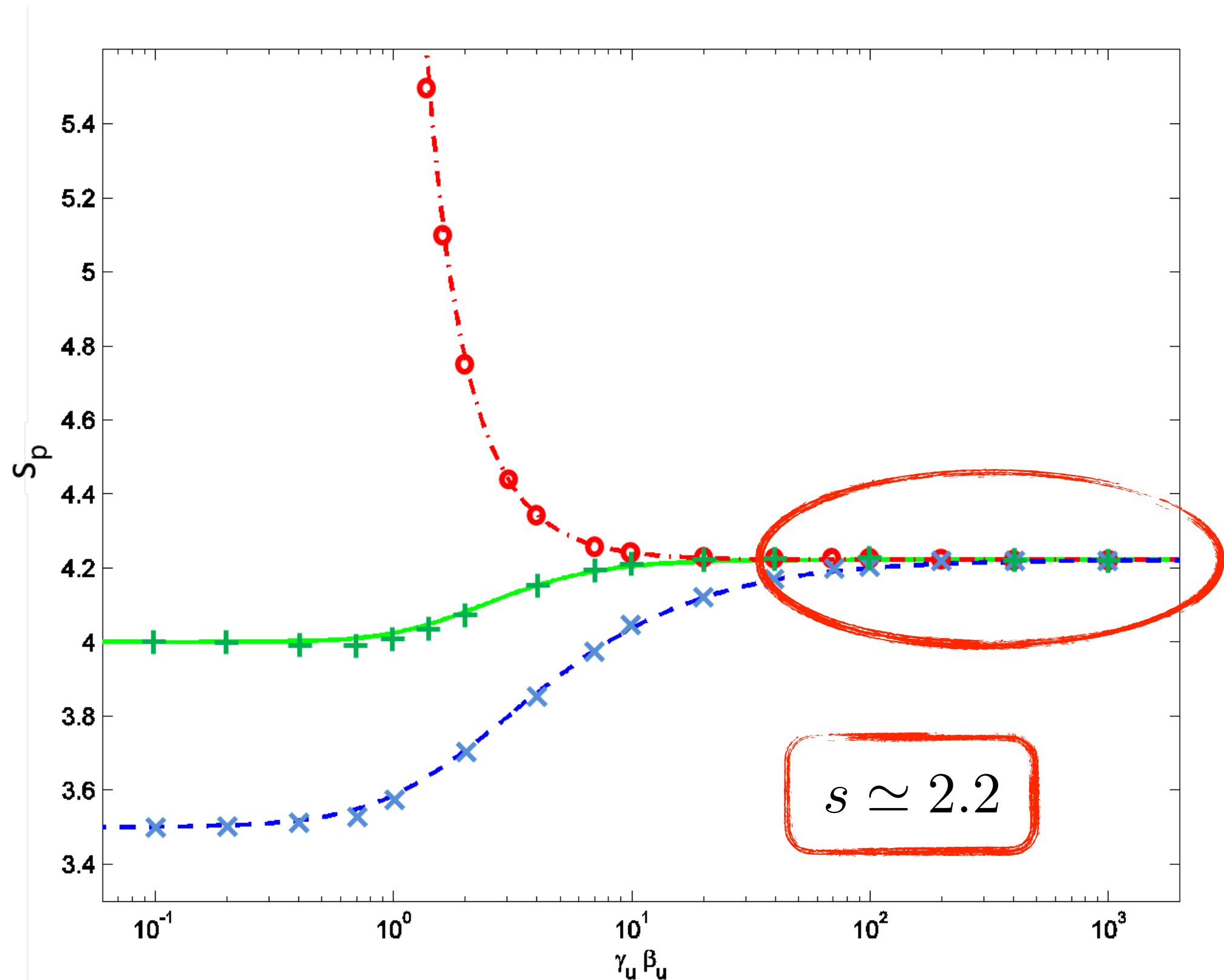


Sironi, Keshet, Lemoine (2015)

Spectrum of accelerated particles

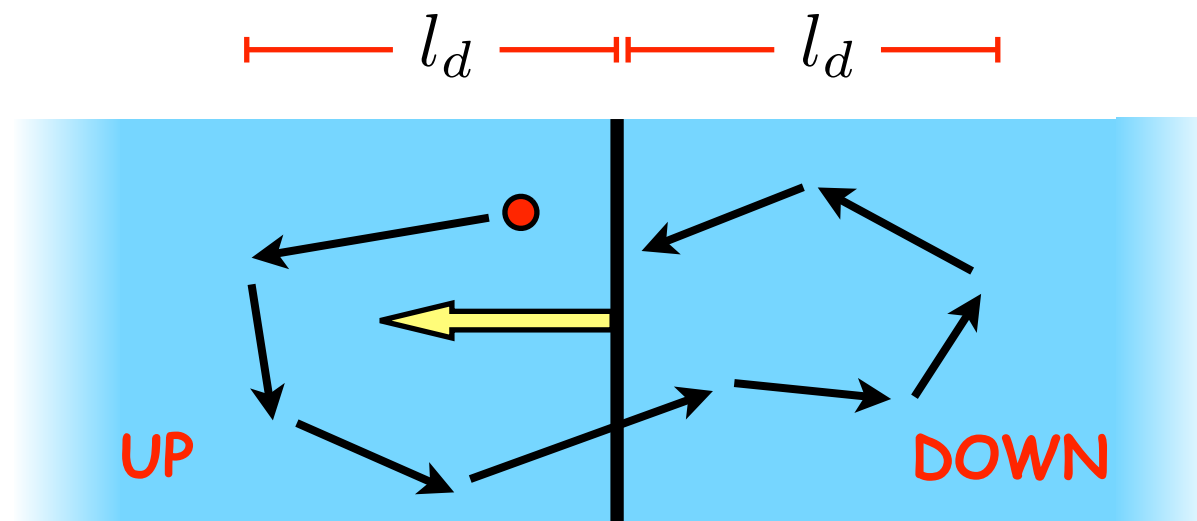


Spectrum of accelerated particles



Key aspects of DSA at ~~non~~-relativistic shocks

particles are accelerated through a series of cycles up→down→up stream



~~non-~~ diffusive transport → ~~isotropic~~

energy gain per cycle → ~~small~~ large

escape probability per cycle → ~~small~~ large

spectral slope → E^{-s}

$$\Delta_{acc} = \Delta E / E \sim \Gamma_s^2 \quad \text{1st crossing}$$

$$\Delta_{acc} = \Delta E / E \sim 1 \quad \text{all others}$$

$$P_{esc} \approx 0.5$$

$$s \simeq 2.2$$

Acceleration time

simple consideration

since $\Delta_{acc} = \Delta E / E \sim 1$ then $\tau_{acc} \sim \tau_{cyc} = t_{up} + t_{down}$

Acceleration time

simple consideration

since $\Delta_{acc} = \Delta E / E \sim 1$ then $\tau_{acc} \sim \tau_{cyc} = t_{up} + t_{down}$

Particles downstream of the shock have a large probability to never come back (acceleration is difficult!), but we consider here the most optimistic condition, i.e., particles for which $t_{down} \ll t_{up}$

Acceleration time

simple consideration

since $\Delta_{acc} = \Delta E / E \sim 1$ then $\tau_{acc} \sim \tau_{cyc} = t_{up} + t_{down}$

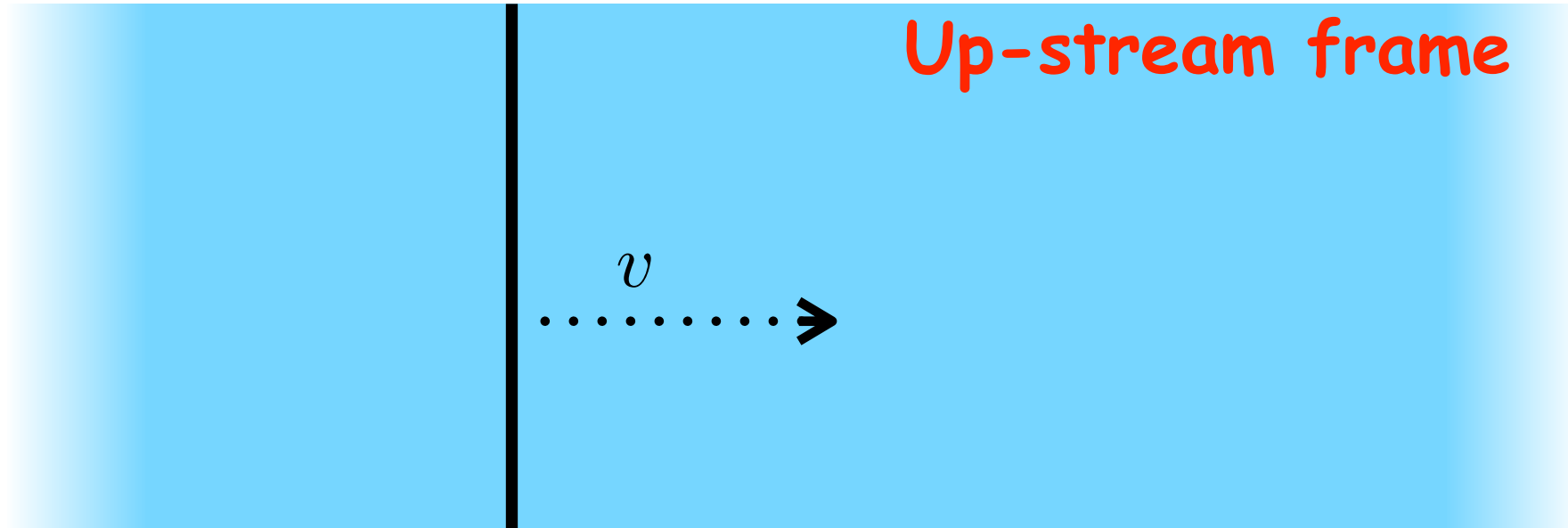
Particles downstream of the shock have a large probability to never come back (acceleration is difficult!), but we consider here the most optimistic condition, i.e., particles for which $t_{down} \ll t_{up}$

$$\tau_{acc} \approx t_{up}$$

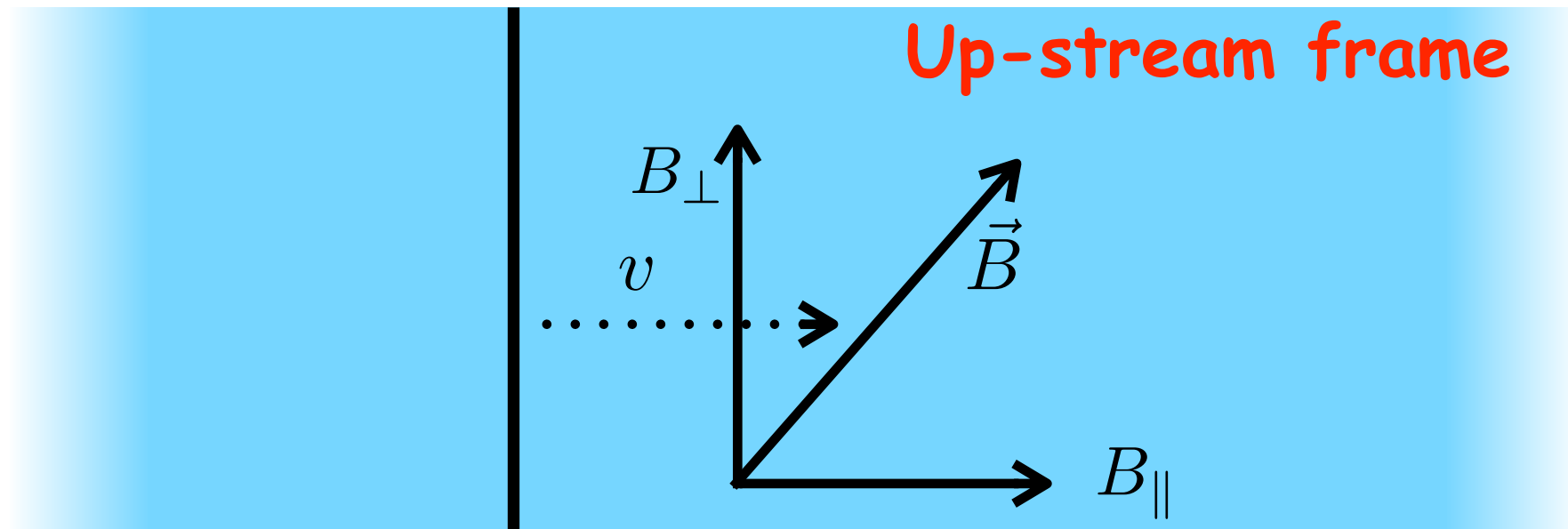
Residence time upstream



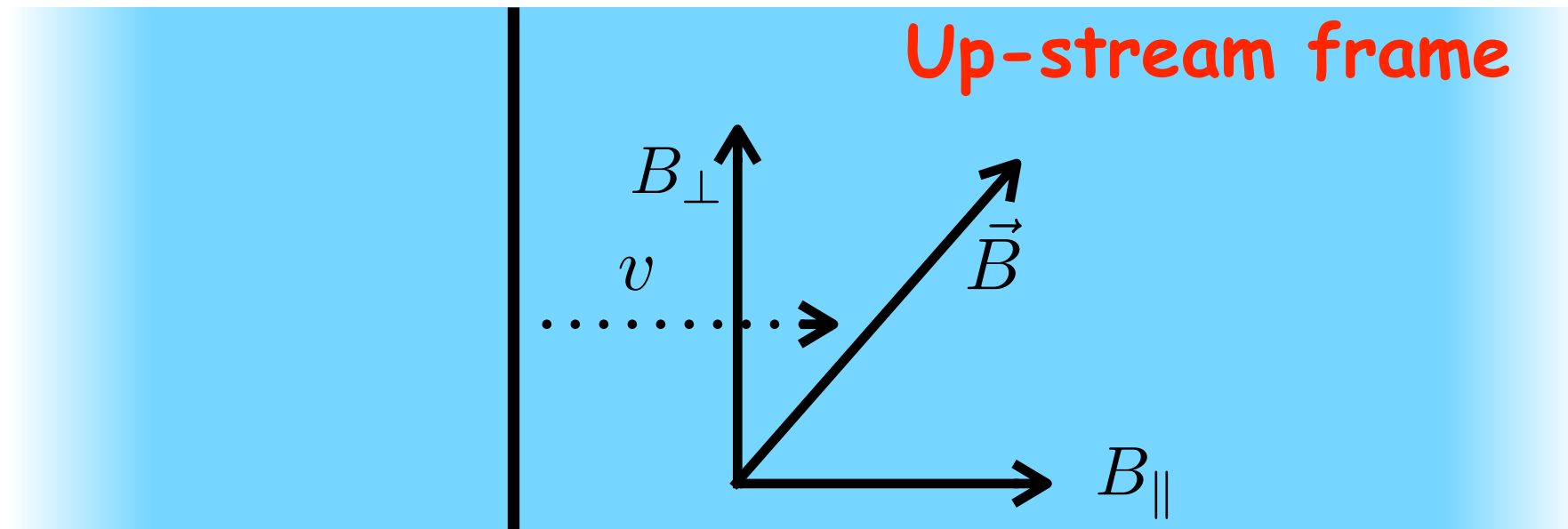
Residence time upstream



Residence time upstream



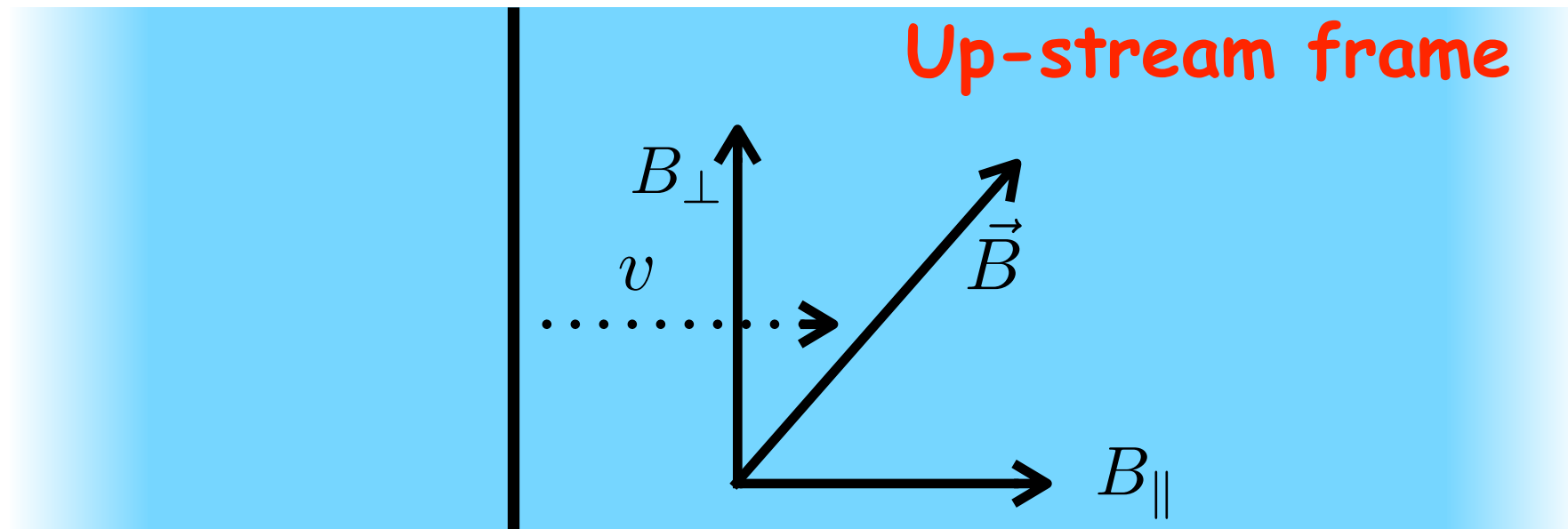
Residence time upstream



Larmor radius

$$R_L \sim \frac{p}{qB_{\perp}} = \frac{\gamma m v}{qB_{\perp}}$$

Residence time upstream

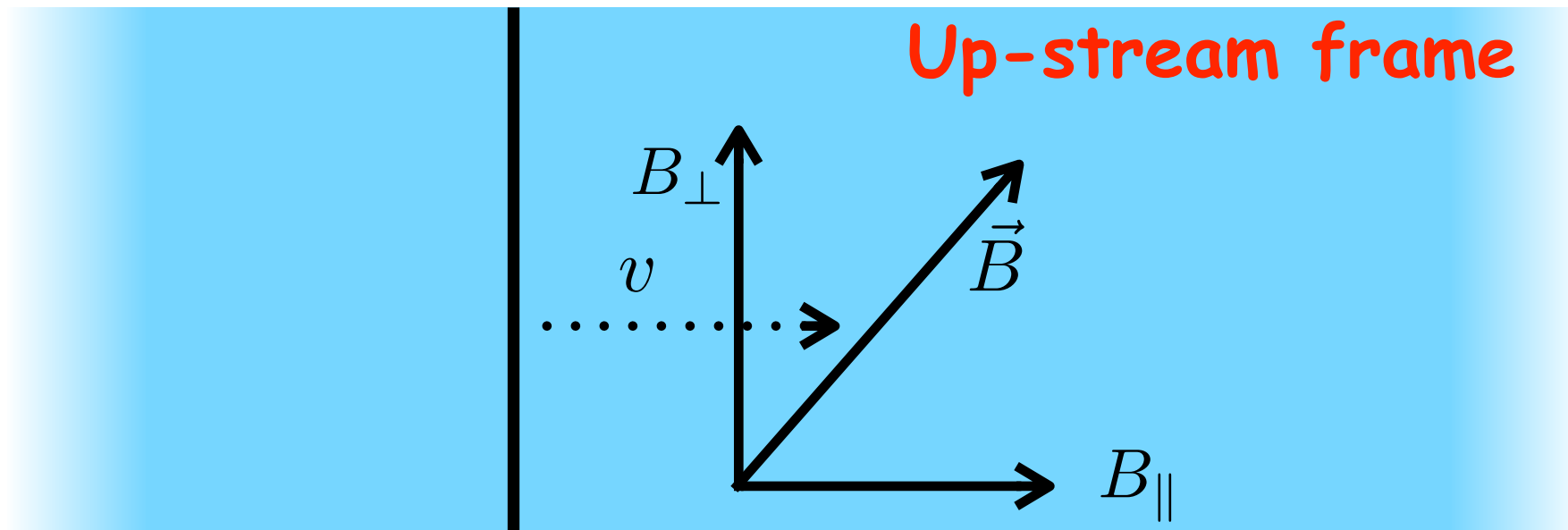


Larmor radius

$$R_L \sim \frac{p}{qB_{\perp}} = \frac{\gamma m v}{qB_{\perp}}$$

a particle is overrun by the shock when it is deflected by an amount $\Delta\theta \sim 1/\Gamma_s$

Residence time upstream



Larmor radius

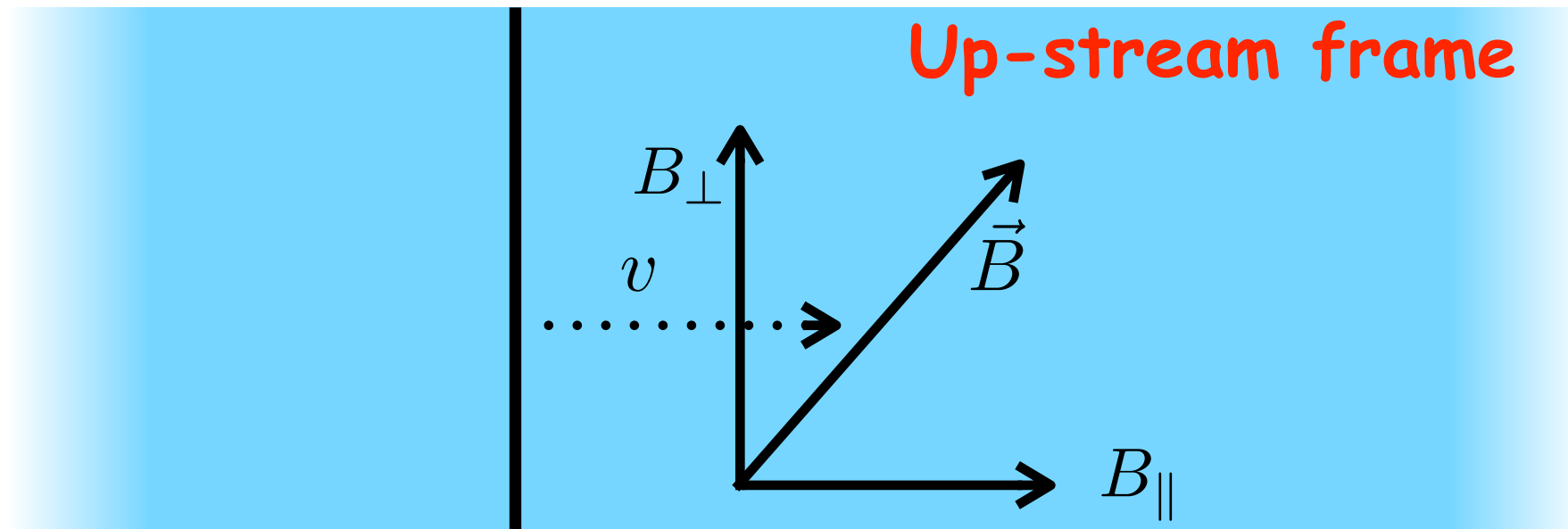
$$R_L \sim \frac{p}{qB_{\perp}} = \frac{\gamma m v}{qB_{\perp}}$$

a particle is overrun by the shock when it is deflected by an amount $\Delta\theta \sim 1/\Gamma_s$

gyration time $\rightarrow$

$$\tau_L = \frac{2\pi R_L}{v} = \frac{2\pi \gamma m}{qB_{\perp}} = \frac{2\pi E}{qB_{\perp}}$$

Residence time upstream



Larmor radius

$$R_L \sim \frac{p}{qB_{\perp}} = \frac{\gamma m v}{qB_{\perp}}$$

a particle is overrun by the shock when it is deflected by an amount $\Delta\theta \sim 1/\Gamma_s$

gyration time $\rightarrow$

$$\tau_L = \frac{2\pi R_L}{v} = \frac{2\pi \gamma m}{qB_{\perp}} = \frac{2\pi E}{qB_{\perp}}$$

residence time
upstream $\rightarrow$

$$t_{up} \sim \frac{\Delta\vartheta}{2\pi} \tau_L \sim \frac{1}{\Gamma_s} \frac{E}{qB_{\perp}} = \frac{1}{\Gamma_s \Omega_{\perp}}$$

$\nwarrow$ gyration
frequency

Acceleration rate

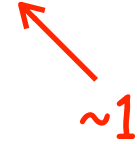
$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E}$$

Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}}$$


A red arrow points from the symbol ~ 1 to the E in the denominator of the second fraction, indicating that $\Delta E/E \sim 1$.

Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}} \sim \Gamma_s \Omega_{\perp}$$


Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}} \sim \Gamma_s \Omega_{\perp}$$


 ~ 1

most optimistic case: no energy losses

Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}} \sim \Gamma_s \Omega_{\perp}$$


 ~ 1

most optimistic case: no energy losses

$$t_{age} \approx \frac{R_s}{c} \longrightarrow t_{age} = t_{up}(E_{max})$$

Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}} \sim \Gamma_s \Omega_{\perp}$$


 ~ 1

most optimistic case: no energy losses

$$t_{age} \approx \frac{R_s}{c} \longrightarrow t_{age} = t_{up}(E_{max})$$

$$\frac{R_s}{c} = \frac{E_{max}}{\Gamma_s q B_{\perp}} \longrightarrow E_{max} \approx \Gamma_s q B_{\perp} R_s$$

Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}} \sim \Gamma_s \Omega_{\perp}$$



most optimistic case: no energy losses

$$t_{age} \approx \frac{R_s}{c} \longrightarrow t_{age} = t_{up}(E_{max})$$

$$\frac{R_s}{c} = \frac{E_{max}}{\Gamma_s q B_{\perp}} \longrightarrow E_{max} \approx \Gamma_s q B_{\perp} R_s$$

non-relativistic shock $\longrightarrow$

$$E_{max} \approx q B u_s R_s$$

Acceleration rate

$$\frac{1}{E} \frac{dE}{dt} \sim \frac{1}{t_{up}} \frac{\Delta E}{E} \sim \frac{1}{t_{up}} \sim \Gamma_s \Omega_{\perp}$$



 ~ 1

most optimistic case: no energy losses

$$t_{age} \approx \frac{R_s}{c} \longrightarrow t_{age} = t_{up}(E_{max})$$

$$\frac{R_s}{c} = \frac{E_{max}}{\Gamma_s q B_{\perp}} \longrightarrow E_{max} \approx \Gamma_s q B_{\perp} R_s$$

non-relativistic shock $\longrightarrow$

$$E_{max} \approx q B u_s R_s$$

Next class



COSMIC RAYS FROM SUPER-NOVAE

BY W. BAADE AND F. ZWICKY

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON AND CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA

Communicated March 19, 1934

A. Introduction.—Two important facts support the view that cosmic rays are of extragalactic origin, if, for the moment, we disregard the possibility that the earth may possess a very high and self-renewing electrostatic potential with respect to interstellar space.

Next class

let's focus on DSA at non-relativistic shocks (it exists!)



COSMIC RAYS FROM SUPER-NOVAE

BY W. BAADE AND F. ZWICKY

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON AND CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA

Communicated March 19, 1934

A. Introduction.—Two important facts support the view that cosmic rays are of extragalactic origin, if, for the moment, we disregard the possibility that the earth may possess a very high and self-renewing electrostatic potential with respect to interstellar space.

Next class

let's focus on DSA at non-relativistic shocks (it exists!)



COSMIC RAYS FROM SUPER-NOVAE

BY W. BAADE AND F. ZWICKY

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON AND CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA

Communicated March 19, 1934

A. Introduction.—Two important facts support the view that cosmic rays are of extragalactic origin, if, for the moment, we disregard the possibility that the earth may possess a very high and self-renewing electrostatic potential with respect to interstellar space.

and try to see if Zwicky was right!