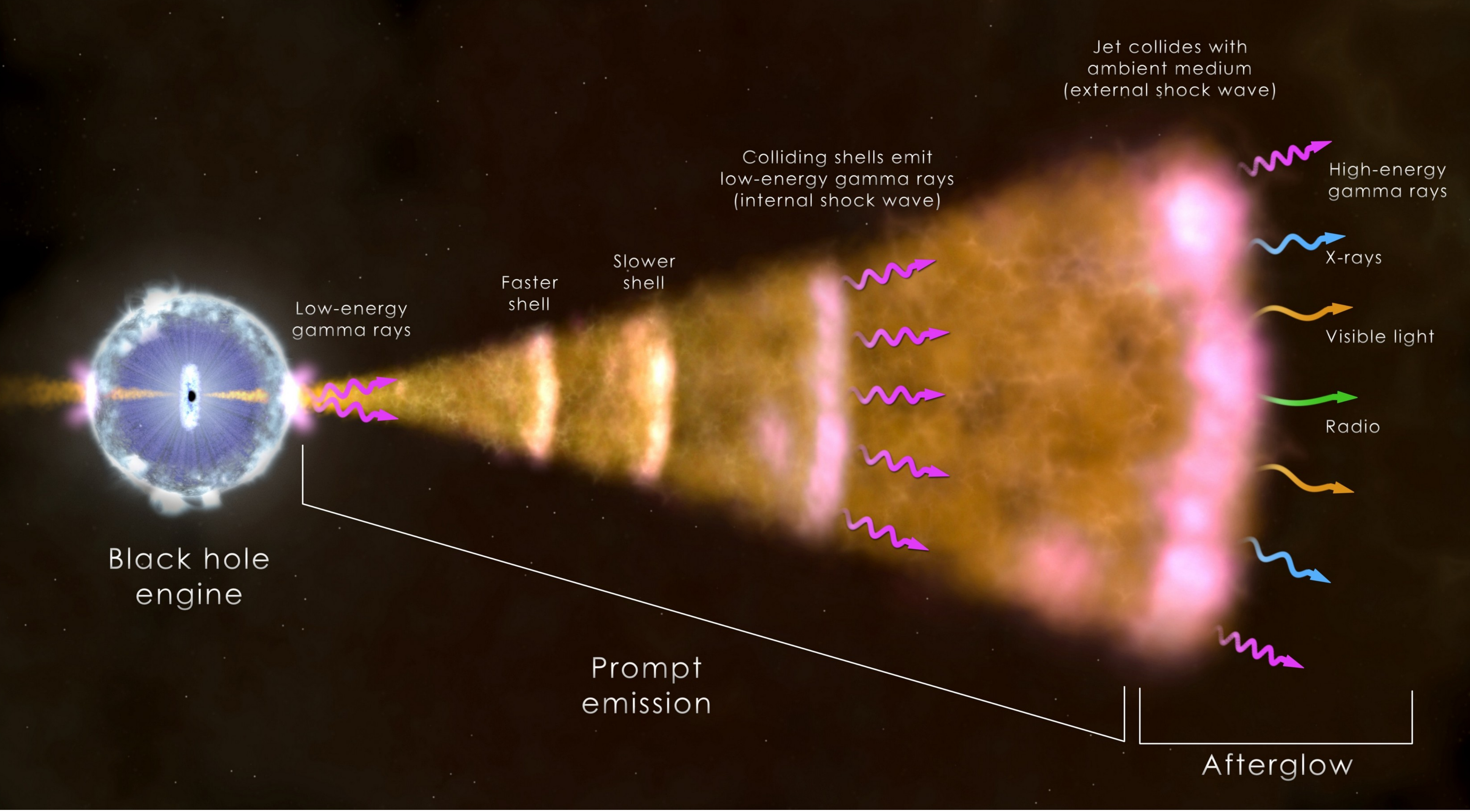
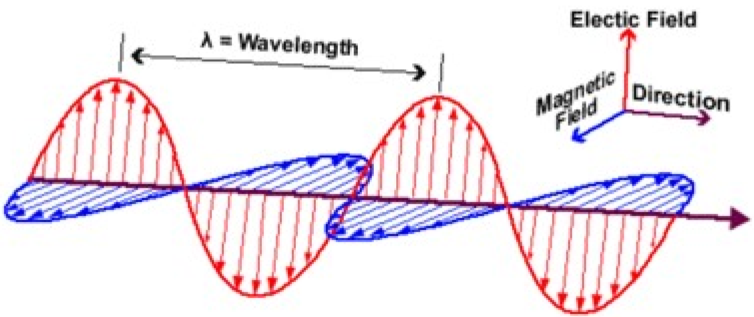



# Thomson & Inverse Compton Scattering

# Thomson Scattering

Thomson Scattering is a process by which an electromagnetic wave is scattered in to random directions by a free electron. It is applicable when:
1. $h\nu \ll mc^2$, with m the electron mass and
2. in the non-relativistic regime (v << c)

First of all, let's consider a linearly polarized e.m. wave incident on a free electron.



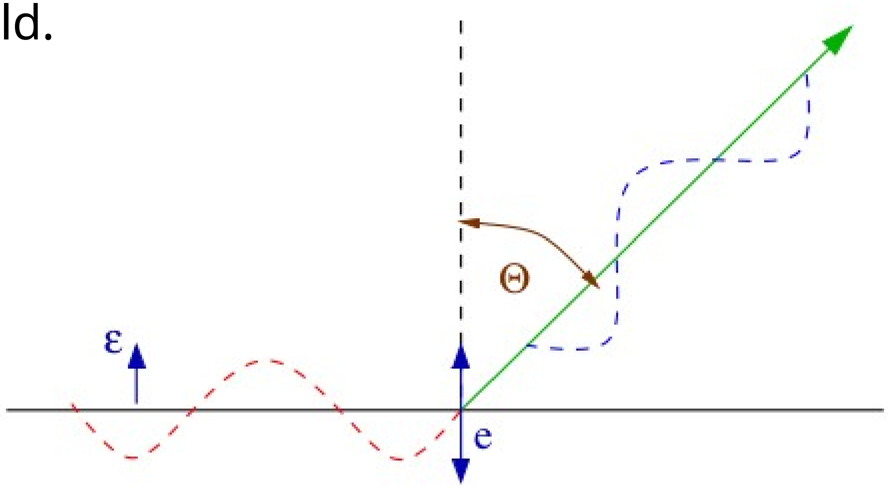

*Linearly polarized wave*

*Which forces act on the electron as the e.m. interacts with it?*

We are left with the electric force:

$$F = e\,\boldsymbol{\epsilon}\,E_0 \sin(\omega_0 t)$$ *force of a linearly polarized wave acting on an electron*

Here $\boldsymbol{\epsilon}$ defines the direction of the **E** field.
We can rewrite this force as:




$$m\ddot{\mathbf{r}} = e\boldsymbol{\epsilon} E_0 \sin \omega_0 t$$

Let's recall the definition of a dipole moment: $\boldsymbol{d} = e\,\boldsymbol{r}$

Therefore:

$$\ddot{\mathbf{d}} = \frac{e^2 E_0}{m} \boldsymbol{\epsilon} \sin \omega_0 t,$$

And: $$\mathbf{d} = -\left(\frac{e^2 E_0}{m\omega_0^2}\right)\boldsymbol{\epsilon} \sin \omega_0 t$$

Now remember what we said in the previous lecture about the Dipole Approximation:

# Dipole Approximation

Following the same procedure as for the single particle case, we can find the total power emitted by an ensemble of particles (in the **non-relativistic** limit) in the so-called *dipole approximation:*

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2\Theta$$

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}.$$

*We can therefore calculate the time average power from* $\ddot{\mathbf{d}} = \frac{e^2 E_0}{m} \boldsymbol{\epsilon} \sin \omega_0 t,$

*(and remembering that the <sin^2(x)> = 1/2)*

This gives:

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \Theta$$

$$P = \frac{e^4 E_0^2}{3m^2 c^3},$$

Remembering that the time averaged Poynting flux is defined as: $\langle S \rangle = \frac{c}{8\pi} E_0^2$

we can write the power as:

$$\boxed{\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi} \frac{d\sigma}{d\Omega}}$$

where we have defined the differential cross section $d\sigma$ for scattering into $d\Omega$ for a polarized electromagnetic wave:

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{polarized}} = \frac{e^4}{m^2 c^4} \sin^2 \Theta = r_0^2 \sin^2 \Theta$$

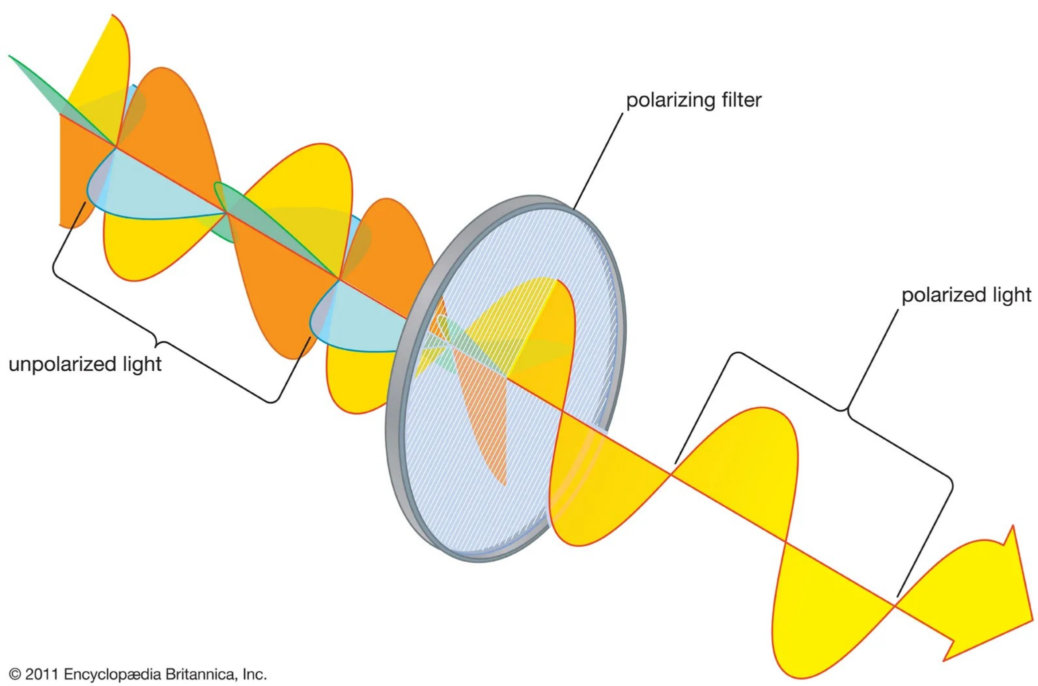
*(classical electron radius)*

To find the total cross section we integrate over the whole solid angle and we get:

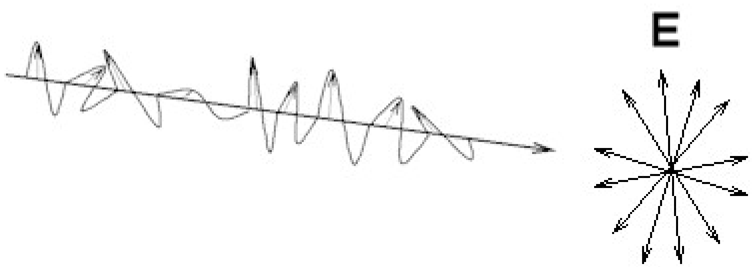
$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1-\mu^2) d\mu. = \frac{8\pi}{3} r_0^2$$

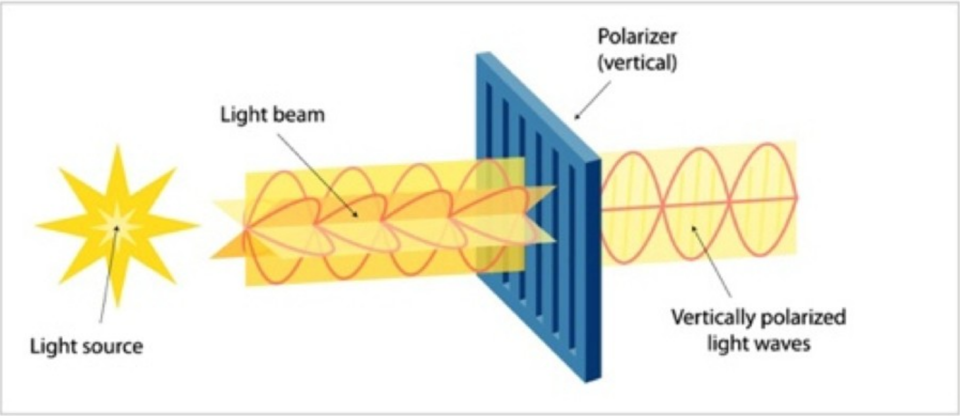
*This one above is the Thomson cross section for an electron and polarized e.m. waves. Note that also the outgoing e.m. wave is polarized (in the plane defined by* $\boldsymbol{\epsilon}$ *and* $\boldsymbol{n}$*)*

What about unpolarized radiation?
"unpolarized radiation can be defined as the superposition of two linearly polarized waves *with perpendicular axis*".



*Unpolarized wave*




An unpolarized electromagnetic wave traveling in the x-direction is a superposition of many waves. For each of these waves the electric field vector is perpendicular to the x-axis, but the angle it makes with the y-axis is different for different waves. For a polarized electromagnetic wave traveling in the x-direction, the angle the electric field makes with the y-axis is unique. Natural light is, in general, unpolarized. The direction of the electric field changes too quickly to be measured.

In other words: unpolarized radiation is a misnomer, since light in this state is composed by a rapidly varying succession of different polarization states. Perhaps a better name would be *randomly polarized* light.

Mathematically this wave can be represented by two orthogonal linearly polarized waves of equal amplitude varying *incoherently.* The word "incoherent" means that the sinusoidal dependence (coherence) of the electric field is lost.

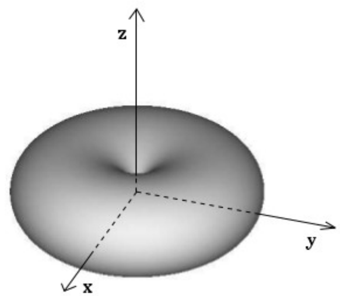
*Differential Thomson cross section for unpolarized radiation:*

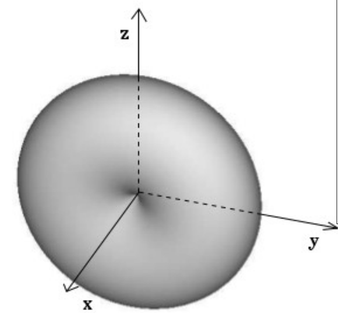
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right]$$
$$= \frac{1}{2} r_0^2 (1 + \sin^2\Theta)$$
$$= \frac{1}{2} r_0^2 (1 + \cos^2\theta),$$

What shape is this?

*Remember what we said yesterday: the power emitted per unit solid angle by an accelerating charge depends as the square of the sinusoid of the angle between the direction of the acceleration and the direction of propagation of radiation*

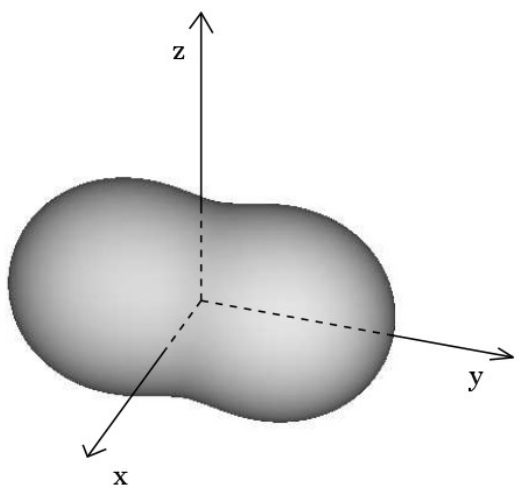
# Pattern of Scattered Radiation (Thomson Electron Scattering)




$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} = \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\rm pol} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\rm pol}\right]$$
$$= \frac{1}{2} r_0^2 (1 + \sin^2\Theta)$$
$$= \frac{1}{2} r_0^2 (1 + \cos^2\theta),$$

*(Try to plot at home the function sin^2(theta) and 1 + sin^2(theta) in polar coordinates as an exercise).*




Note that after the integration:

$$\sigma_{\rm unpol} = \sigma_{\rm pol} = (8\pi/3) r_0^2.$$

Also the scattered radiation will be polarized with a certain degree:

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta}$$

# Electron scattering (Thomson) optical depth

*Remember that:* $\alpha_\nu = n\sigma_\nu$.

Here sigma is the Thomson scattering cross section (units: cm^2)
n is the number density of particles.

The Thomson cross section has a value of 6.65e-25 cm^2

Let's go back to our problem of the Orion Nebula (Lecture 5).
We said n~10,000 and R=10^19 cm.

*Therefore the optical depth due to electron scattering is:*

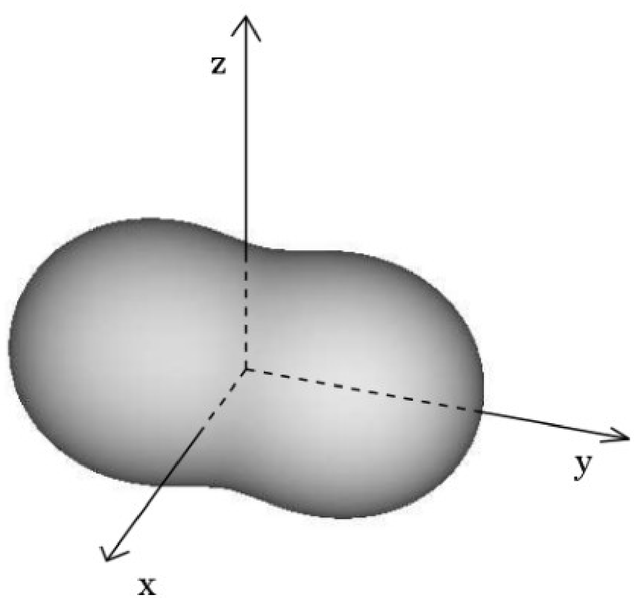
$$\tau = n\sigma_T R \approx 0.07$$

*Thus our HII cloud is optically thin both for electron scattering and for free-free absorption, thus we are really seeing Bremsstrahlung radiation!*

# Recap Thomson Scattering

Recall what we said about Thomson (electron) scattering:

1. It occurs when the photon’s energy is << electron rest mass
2. The electrons move non-relativistically: v<<c.




The radiation pattern has a “peanut shape”. The incoming and outgoing photon has the same energy and the electron does not change energy in the scattering process (elastic or coherent scattering).

# Photon scattering by electrons - Overview

| | **Low energy photons** | **High energy photons** |
| --- | --- | --- |
| $v \ll c$ | $\hbar\omega \ll m_e c^2$<br>Thomson scattering<br>Classical treatment<br>frequency unchanged | $\hbar\omega \geq m_e c^2$<br>Compton scattering<br>Quantum treatment incorporating photon momentum<br>frequency decreases |
| $v \sim c$ | $\gamma\hbar\omega \ll m_e c^2$<br>Inverse Compton<br>Photons gain energy from relativistic electrons<br>Approximate with classical treatment in electron rest frame<br>Frequency increases | $\gamma\hbar\omega \geq m_e c^2$<br>Inverse Compton<br>Quantum treatment in electron rest frame<br>Photons gain energy from relativistic electrons<br>Frequency increases |

Preamble on the notation:

– The prime symbol ' means that the quantity is calculated in the rest frame K' (i.e., the electron's rest frame in this case)

– No prime symbol means the quantity is calculated in the lab frame K (i.e., observer frame).

– The under-script 1 means that the quantity is calculated after the scattering has already occurred.

– No under-script means that the quantity is calculated before the scattering.

E.g.: $E$ → energy before the scattering in K
$E_1$ → energy after the scattering in K
$E'$ → energy before the scattering in K'
$E'_1$ → energy after the scattering in K'

# Direct Compton

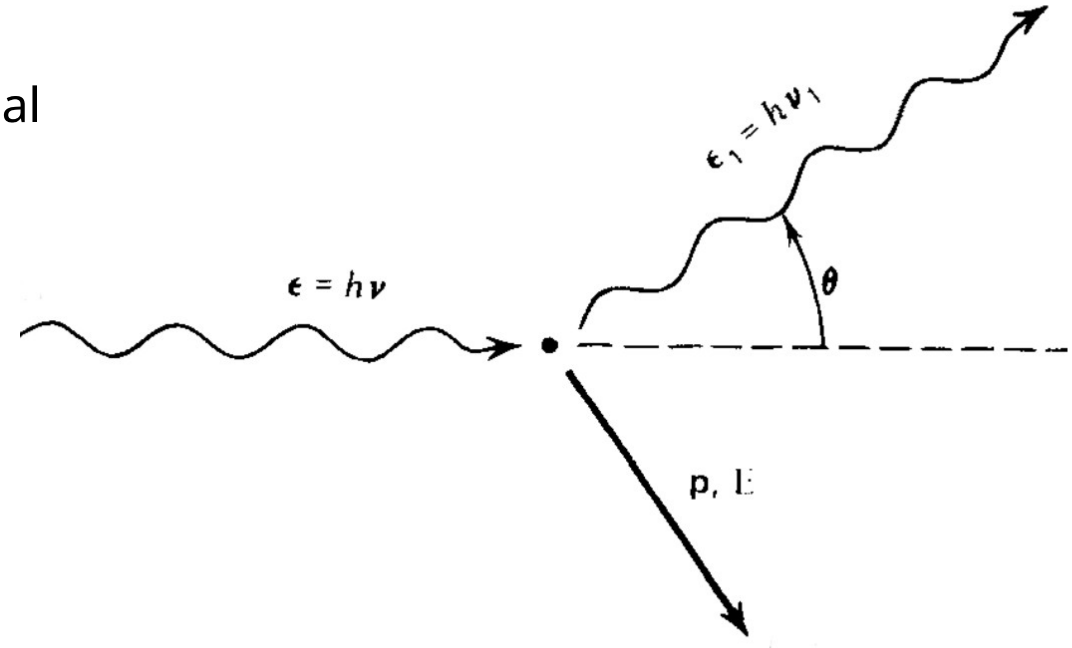
Let’s start by looking at the momentum and energy of the photon and electrons.
In Thomson scattering the photon has no momentum (classical electrodynamics).
However, from quantum mechanics we do know that a photon has a momentum.
This means that any scattering process cannot be purely elastic since the electron will recoil due to the momentum of the photon.

The photon has initial energy $\epsilon$ and final energy $\epsilon_1$.

The photon has initial momentum $\epsilon/c$ and final momentum $\epsilon_1/c$ .

The **electron** has initial energy mc^2 and final energy E/c.

The electron has initial momentum 0 and final momentum p.



# Direct Compton

Using the conservation of energy and momentum it's easy to show that the final and initial photon energies are related in the following way:

$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1 - \cos\theta)}$$

***Direct Compton Scattering***

This can be rewritten in terms of wavelengths as:

$$\lambda_1 - \lambda = \lambda_c(1 - \cos\theta)$$

where the subscript "c" refers to the Compton wavelengt

$$\lambda_c \equiv \frac{h}{mc}$$

$$= 0.02426 \text{ Å for electrons.}$$

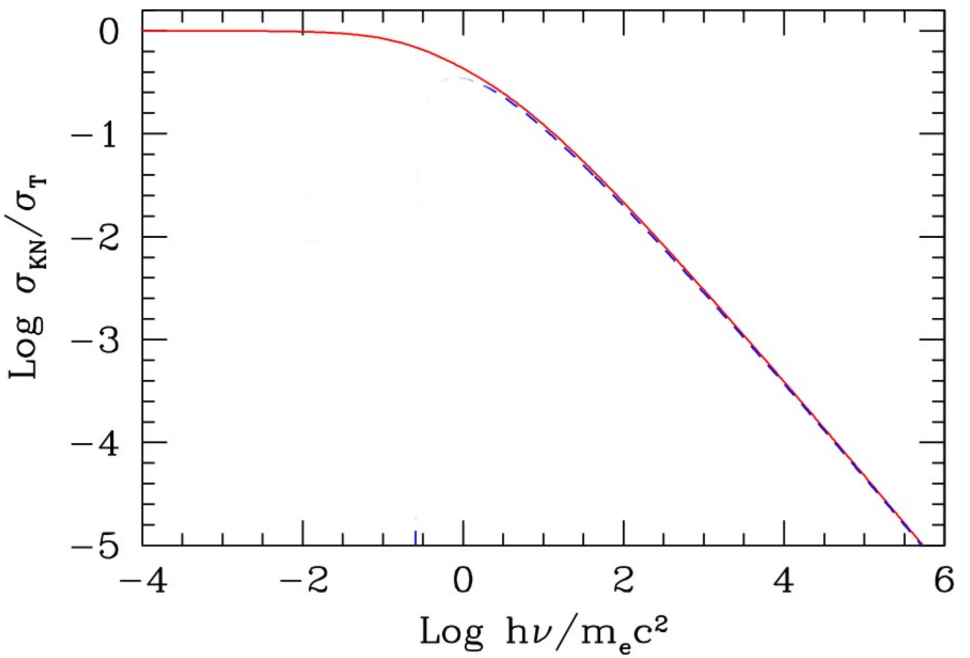
# Klein-Nishina Cross Section

Quantum effects enter in the Compton scattering not only because of the photon momentum. They also change the cross section of the electron.

When the energy of the photon approaches the rest mass energy of the electron, then the Thomson cross section changes and becomes the so-called Klein-Nishina cross section: ( the variable $x$ is defined as the dimensionless photon energy in units of the electron rest mass):

$$\sigma = \sigma_T \cdot \frac{3}{4}\left[ \frac{1+x}{x^3}\left\{ \frac{2x(1+x)}{1+2x} - \ln(1+2x) \right\} + \frac{1}{2x}\ln(1+2x) - \frac{1+3x}{(1+2x)^2} \right]$$

The “take home” message is that the KN cross section is smaller than the Thomson one



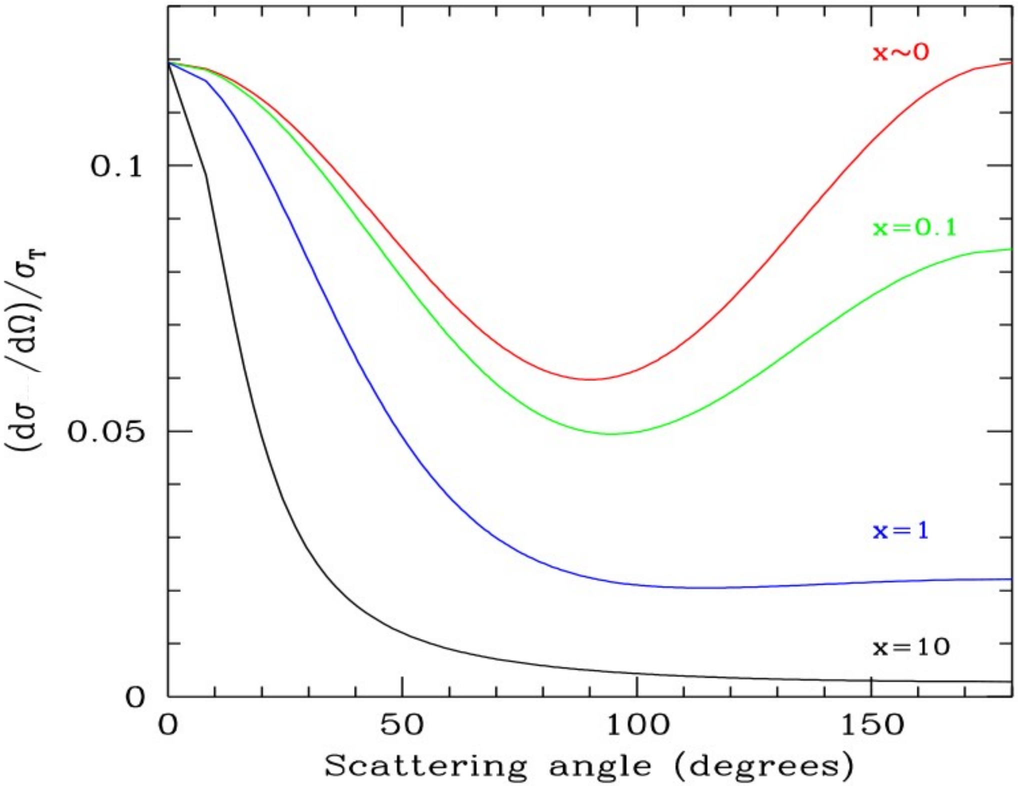
# Klein-Nishina Cross Section

It's important to have a look at the differential KN cross section per unit solid angle, since this is how we understand the radiation pattern of the scattering process:

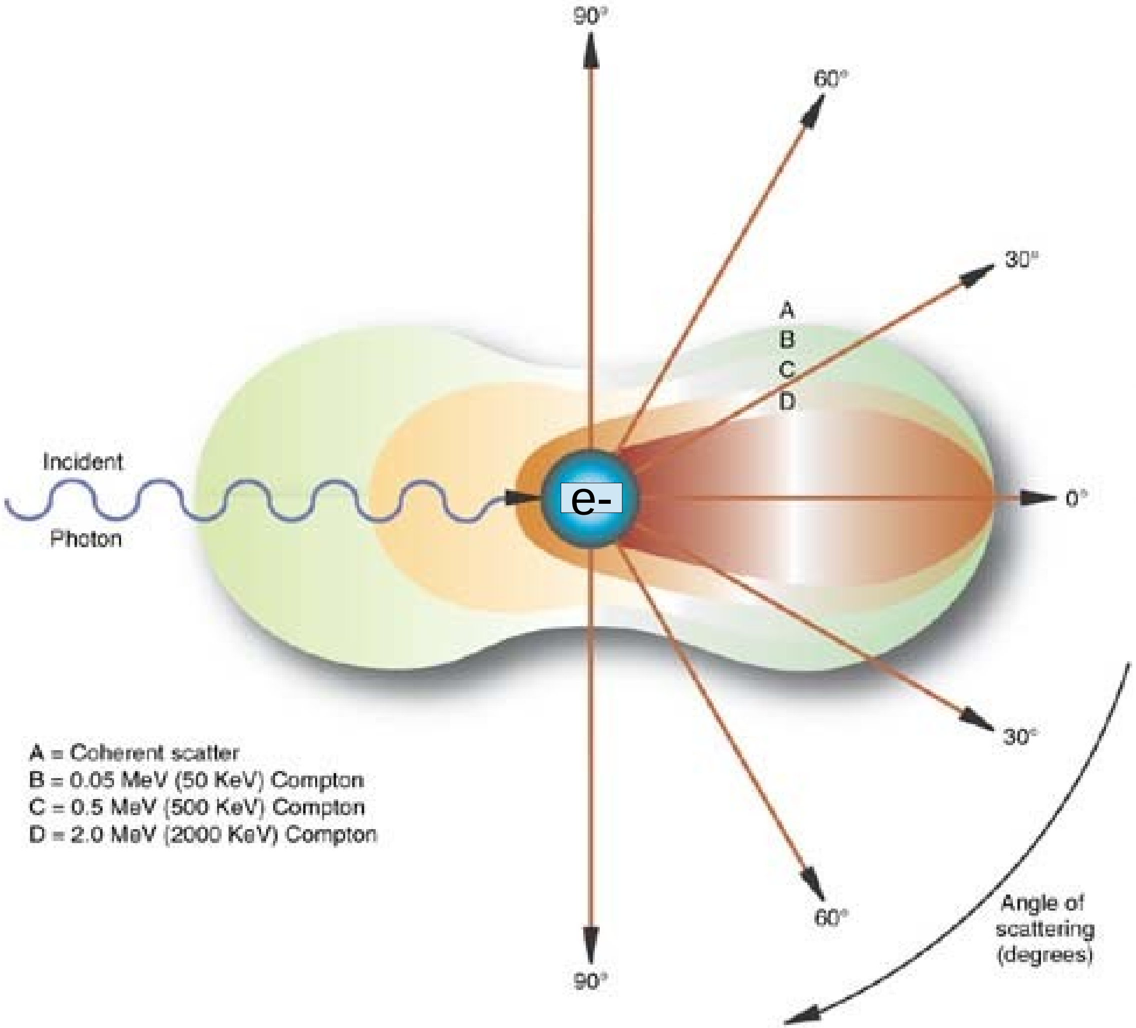
$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon_1}{\epsilon} - \sin^2\theta\right)$$

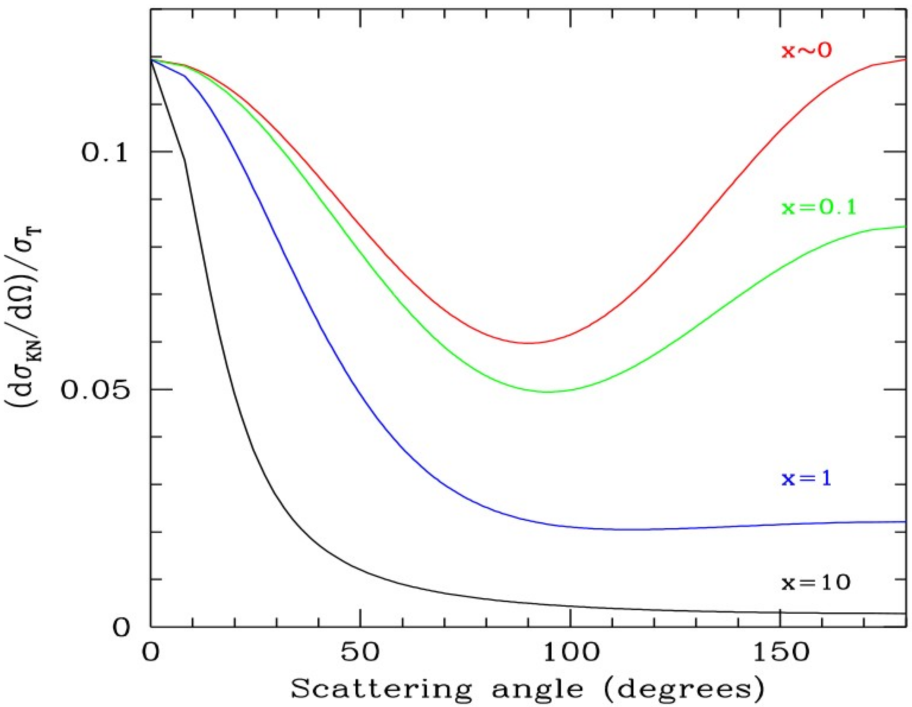
If we plot this function, we see that the scattering is not isotropic but becomes preferentially "forward" as the energy of the photon increases

Note that the x~0 corresponds to the Thomson scattering and the familiar "peanut shape".



# Radiation Pattern



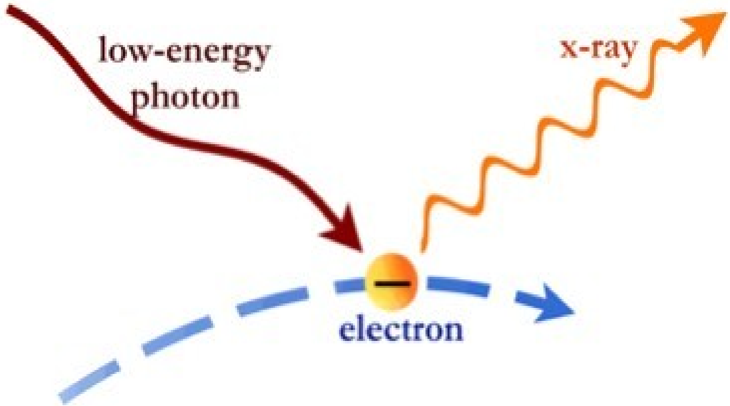




The green “peanut shape” pattern is the Thomson scattering (x~0, coherent scattering)
As the energy is increased the peanut shape disappears and the scattering becomes elongated in the “forward” direction.

# Inverse Compton Scattering

The direct Compton (or simply Compton) scattering is not a very common process in astrophysics, but its inverse process is very widespread: inverse Compton.

This is a relativistic and quantum phenomenon at the same time (whereas direct Compton is a purely quantum phenomenon).



**The first question we would like to answer now is:**

How does the initial photon energy change after a collision with the relativistic electron? (Inverse Compton)

Remember the notation:

– The prime symbol ' means that the quantity is calculated in the rest frame K' (i.e., the electron's rest frame in this case)

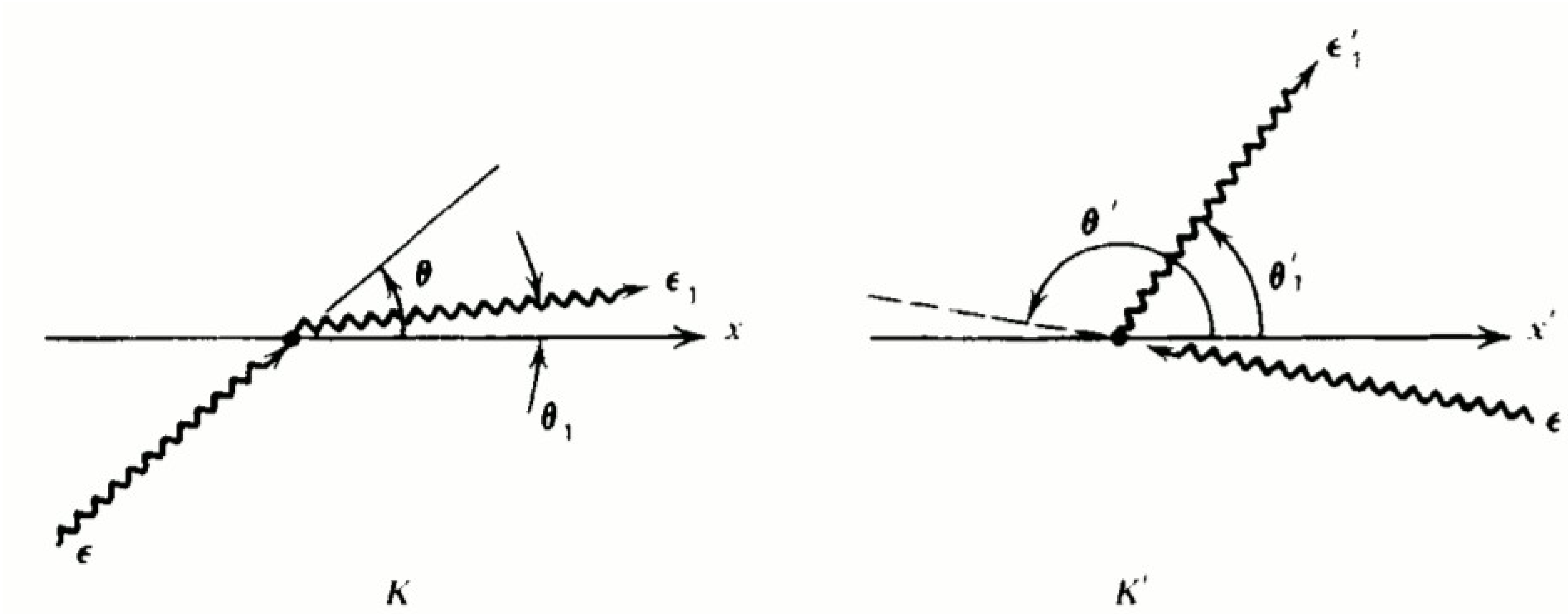
– No prime symbol means the quantity is calculated in the lab frame K (i.e., observer frame).

– The under-script 1 means that the quantity is calculated after the scattering has already occurred.

– No under-script means that the quantity is calculated before the scattering.

E.g.: $E$ → energy before the scattering in K
$E_1$ → energy after the scattering in K
$E'$ → energy before the scattering in K'
$E'_1$ → energy after the scattering in K'

# Inverse Compton

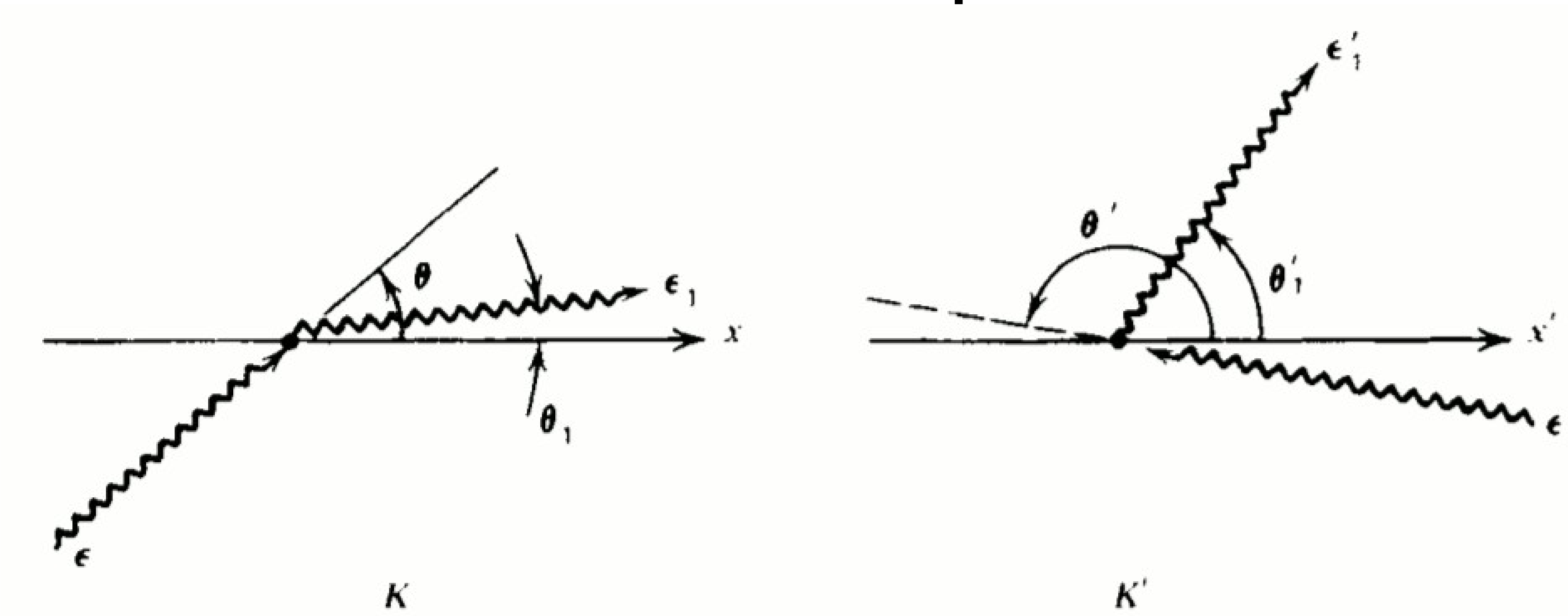



**Step 1**: Photon and e- in lab frame → e- rest frame

**Step 2**: Photon and e- interact in e- rest-frame

**Step 3**: Back to the lab frame

# Inverse Compton



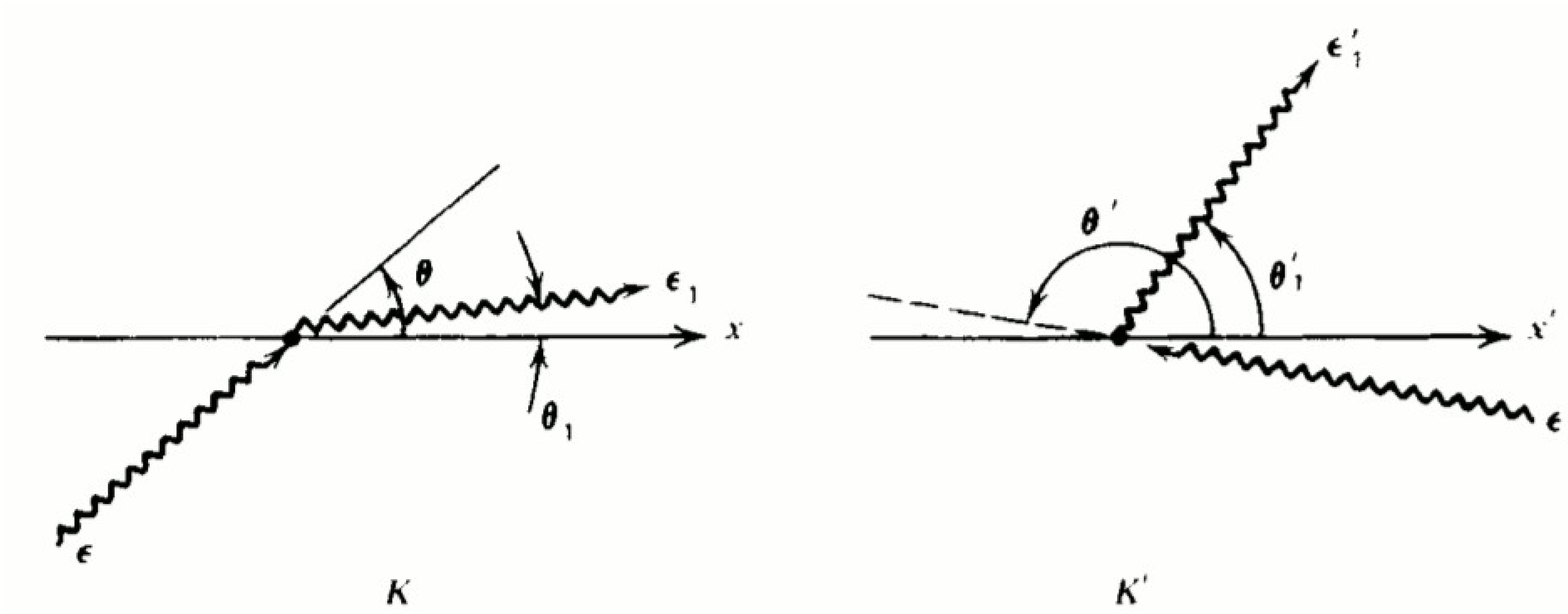

Step 1: lab frame → e- rest frame

$$\epsilon' = \epsilon\gamma(1 - \beta\cos\theta)$$

**epsilon** is the energy of the photon in K
**epsilon'** is the energy of the photon in K'

BEFORE THE SCATTERING (see Eq. 4.12 on Rel. Doppler)
Note: The photon energy has increased in the e- rest frame
(due to the relativistic Doppler boost)

# Inverse Compton

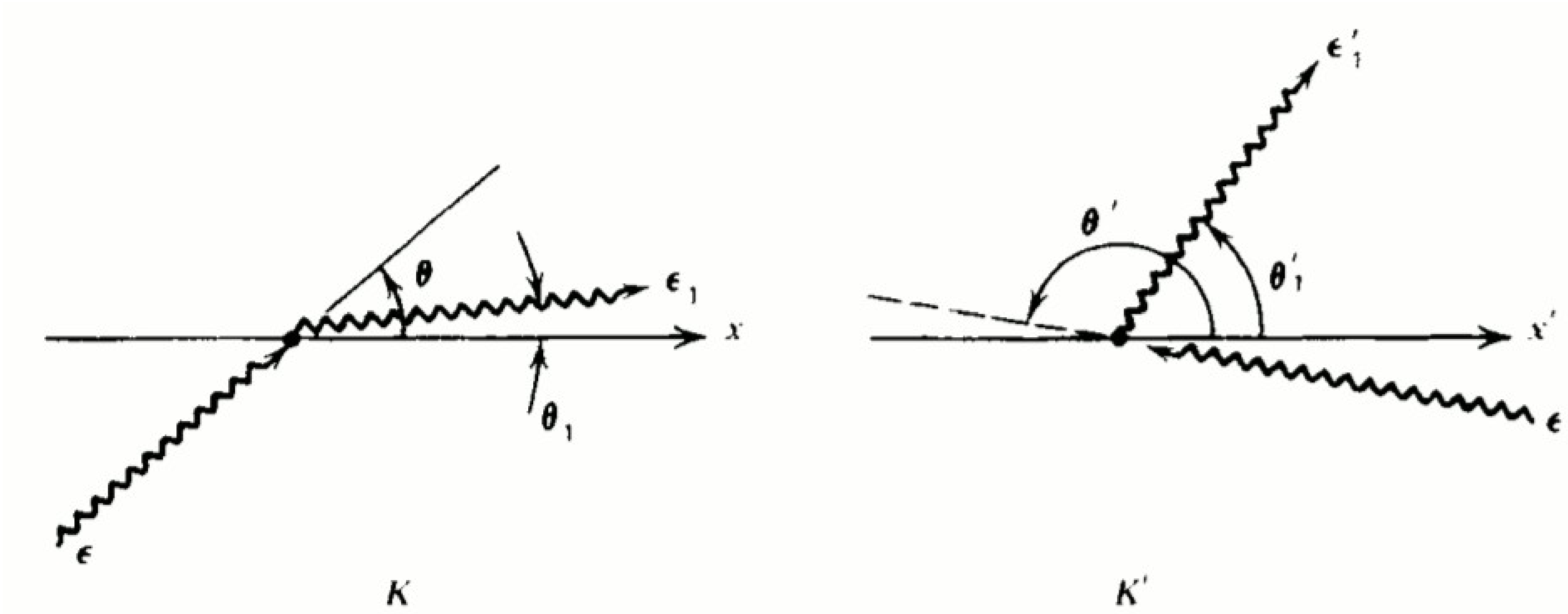



**Step 2**: Photon and e- interact in e- rest-frame

$$\epsilon_1' = \frac{\epsilon'}{1 + \frac{\epsilon'(1-\cos\Theta)}{mc^2}} \approx \epsilon'\left[1 - \frac{\epsilon'}{mc^2}(1-\cos\Theta)\right]$$

**We are now in the e- rest frame, so we can use the normal Compton formula (Eq. 7.2)**

# Inverse Compton



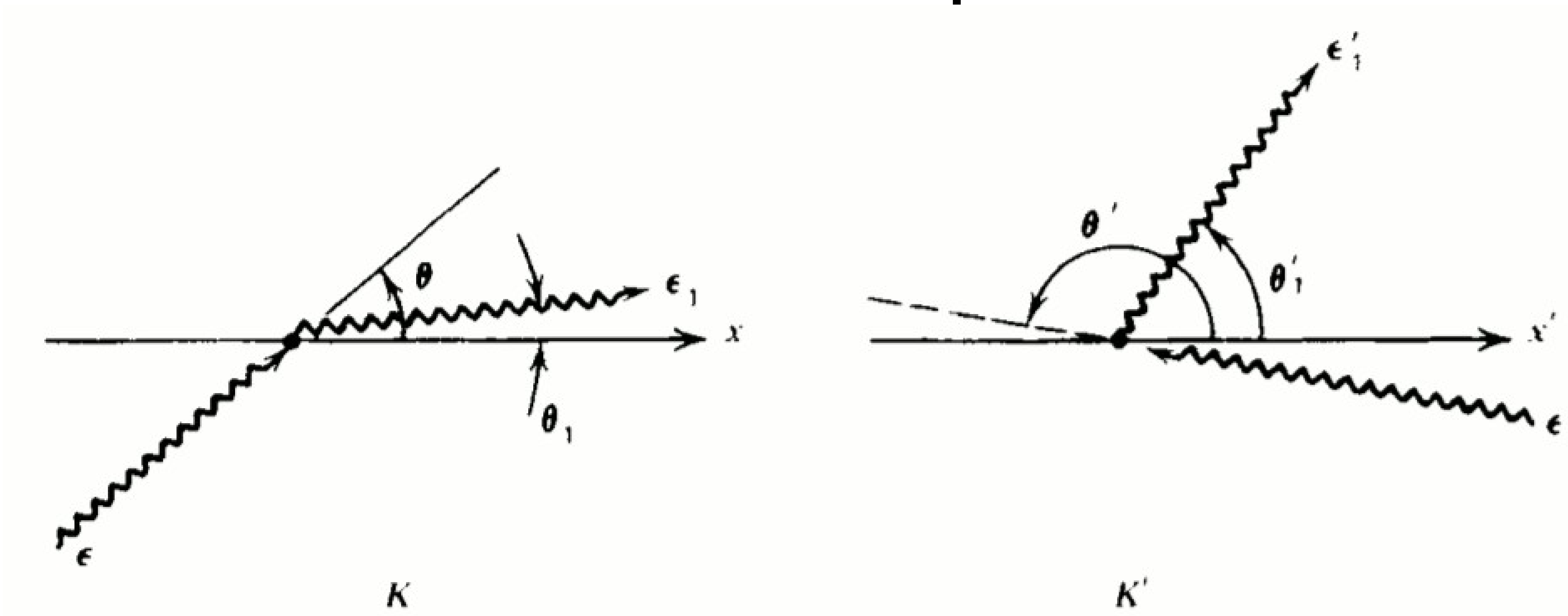

**Step 2**: Photon and e- interact in e- rest-frame

$$\epsilon_1' = \frac{\epsilon'}{1 + \frac{\epsilon'(1-\cos\Theta)}{mc^2}} \approx \epsilon' \left[ 1 - \frac{\epsilon'}{mc^2}(1-\cos\Theta) \right]$$

**Note:** $\cos\Theta = \cos\theta_1' \cos\theta' + \sin\theta' \sin\theta_1' \cos(\phi' - \phi_1')$

Phi is now the azimuthal angle between scattered photon and incident photon in the e- rest

# Inverse Compton

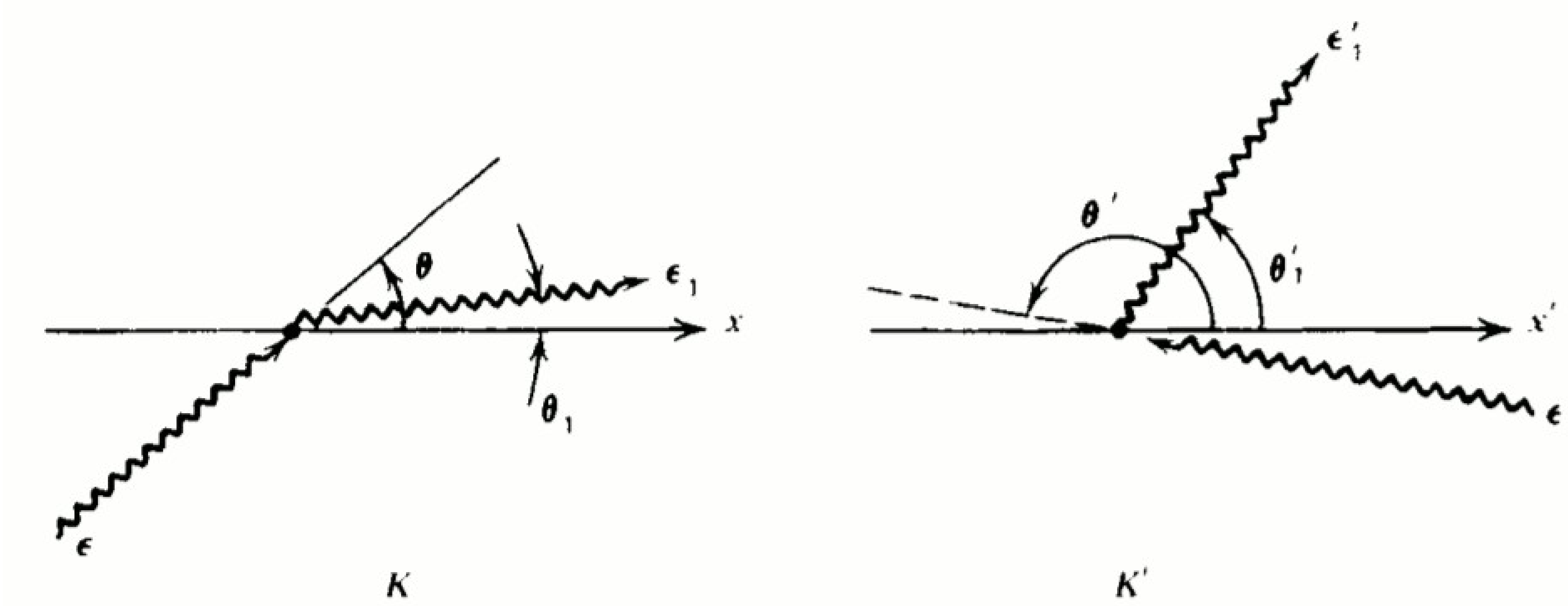



**Step 2**: Photon and e- interact in e- rest-frame

$$\epsilon_1' = \frac{\epsilon'}{1+\frac{\epsilon'(1-\cos\Theta)}{mc^2}} \approx \epsilon'\left[1-\frac{\epsilon'}{mc^2}(1-\cos\Theta)\right]$$

**Note 2: The photon energy has *decreased* in this process since some energy was given away to the electron**

# Inverse Compton

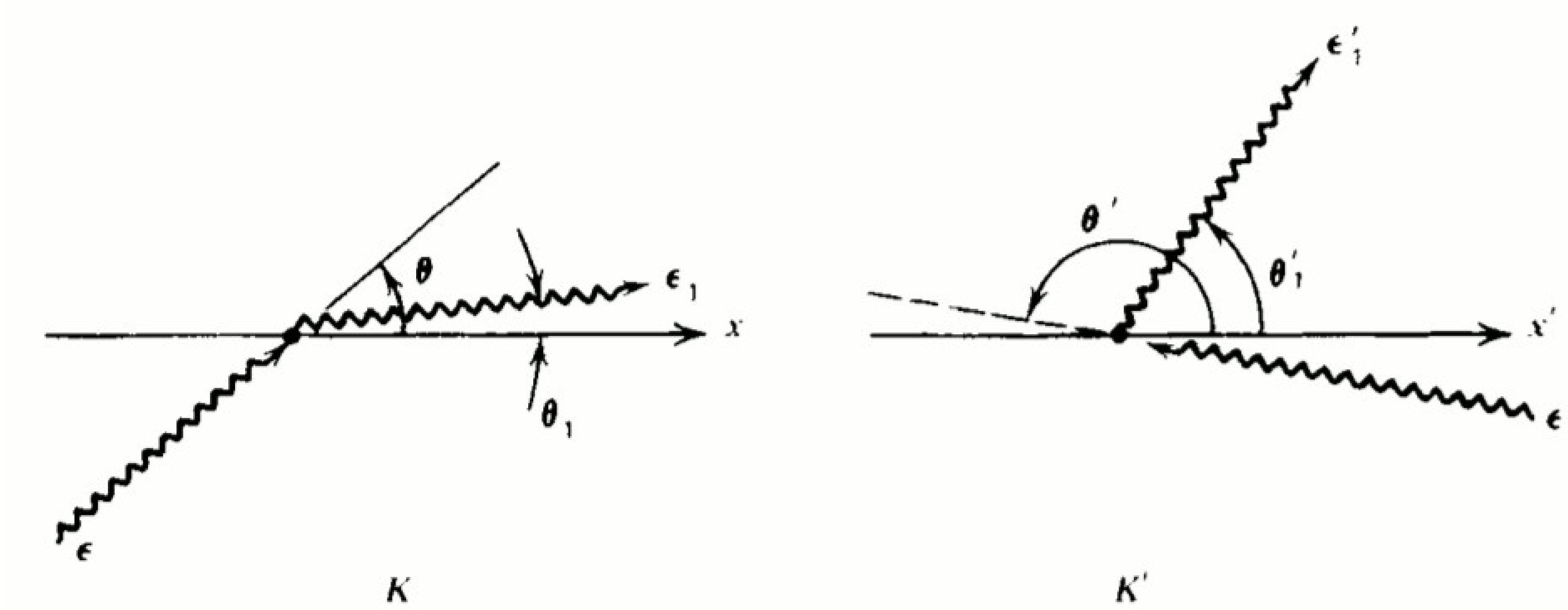



**Step 2**: Photon and e- interact in e- rest-frame

$$\epsilon_1' = \frac{\epsilon'}{1 + \frac{\epsilon'(1-\cos\Theta)}{mc^2}} \approx \epsilon' \left[1 - \frac{\epsilon'}{mc^2}(1-\cos\Theta)\right] \approx \epsilon'$$

**Note 3: If the photon energy is still << mc^2 then we are still in the Thomson regime**

# Inverse Compton




**Step 3**: Back to the lab frame

$$\epsilon_1 = \epsilon_1{}' \gamma \left(1 + \beta \cos \theta_1{}'\right)$$

**Note 4: We have a second Doppler boost because now we go back to the lab frame. The photon energy has *increased* again (second relativistic Doppler boost)**

## Some comments:

**i.** In our step 2 we have used the Thomson scattering, i.e., ***we have assumed*** that the photon energy is << mc^2. In other words in the electron rest frame the scattered photon will have the same energy as before the scattering (E'1 = E').

**ii.** The photon *gains* an energy in step 1 and 3 by a factor gamma (so in total it gains a factor gamma^2 because of the double Relativistic Doppler boost).

**iii.** The photon *loses* energy in step 2 (but this loss is basically ~0 if we're in the Thomson regime).

**Summary:**

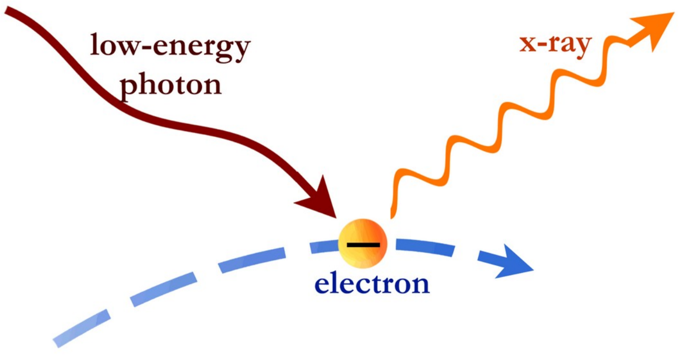
**Step 1. K → K' (1st Rel. Doppler boost)**
**Step 2. Compton Scattering (photon loses energy to the electron)**
**Step 3. K' → (2nd Rel. Doppler boost)**

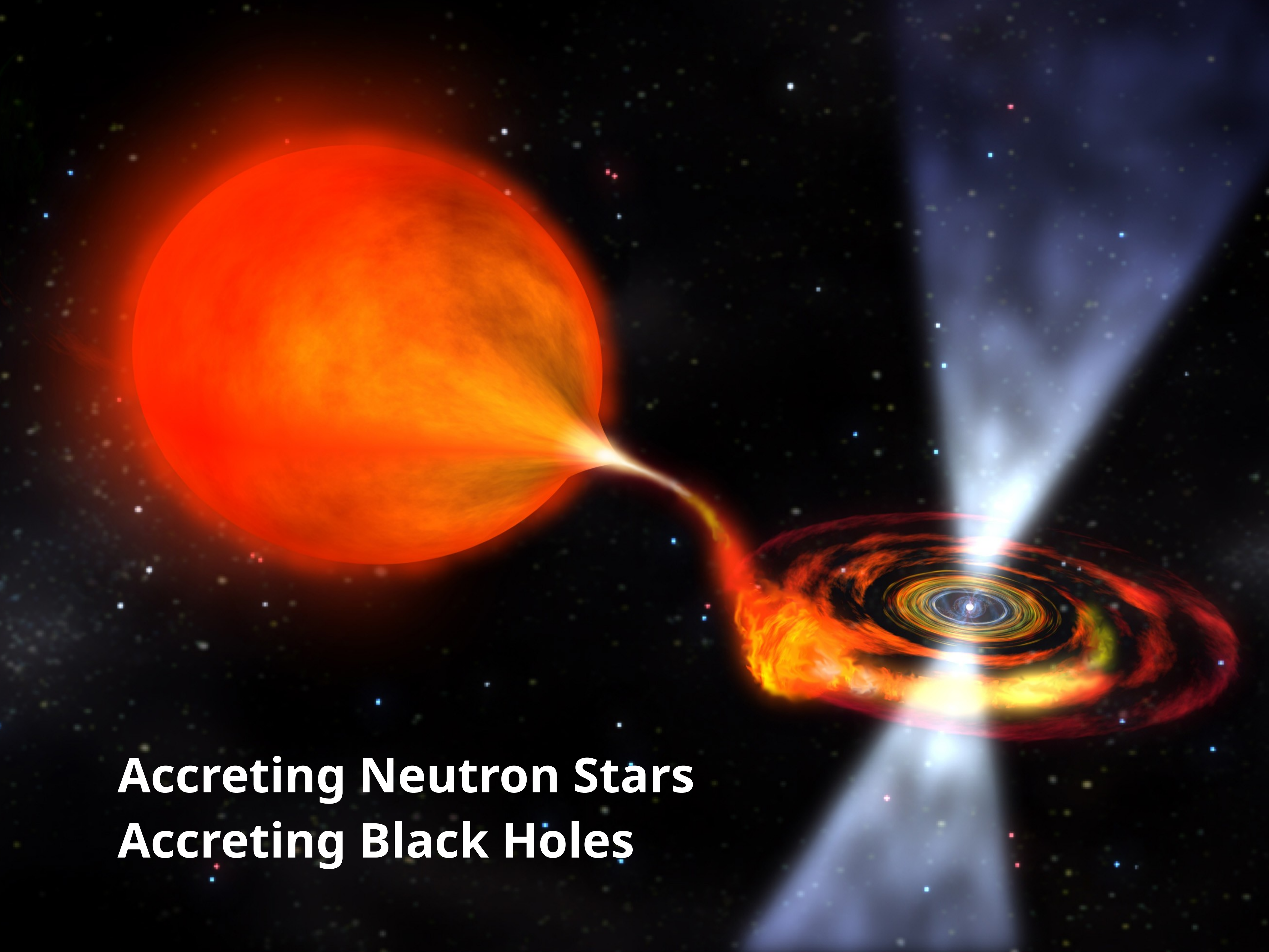
Example:

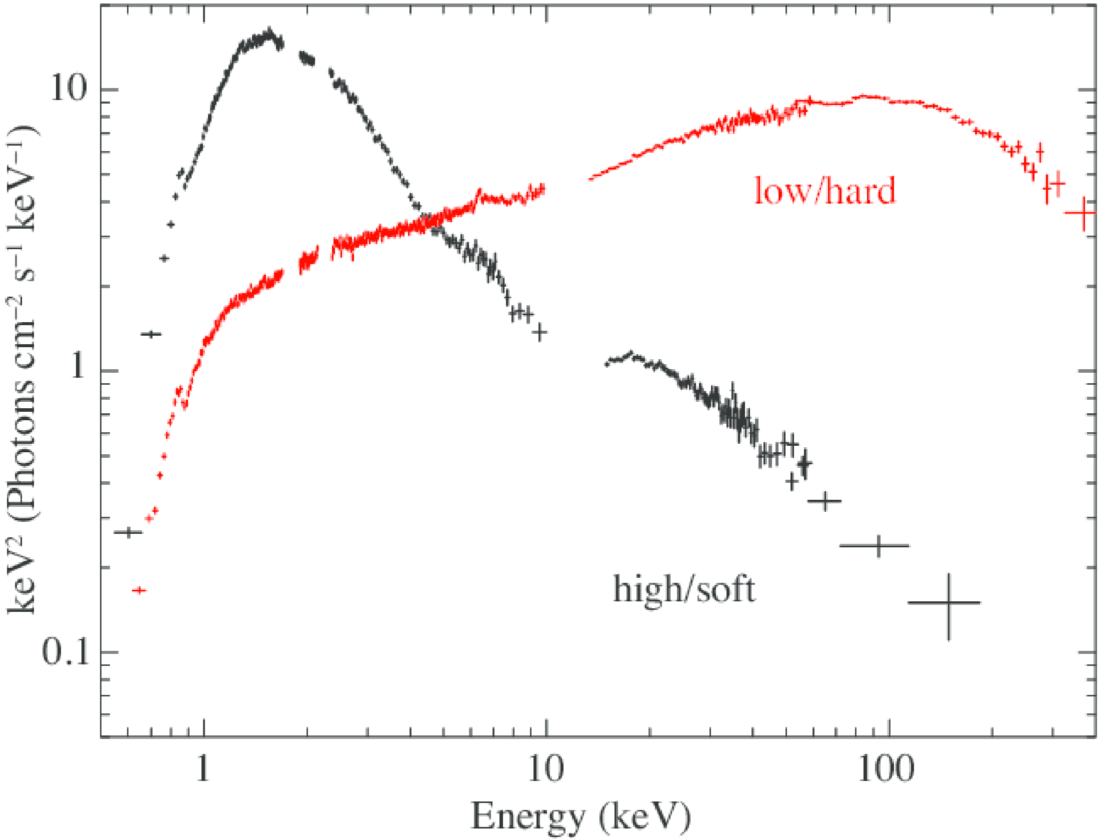
The rest mass energy of an electron is 511 keV.

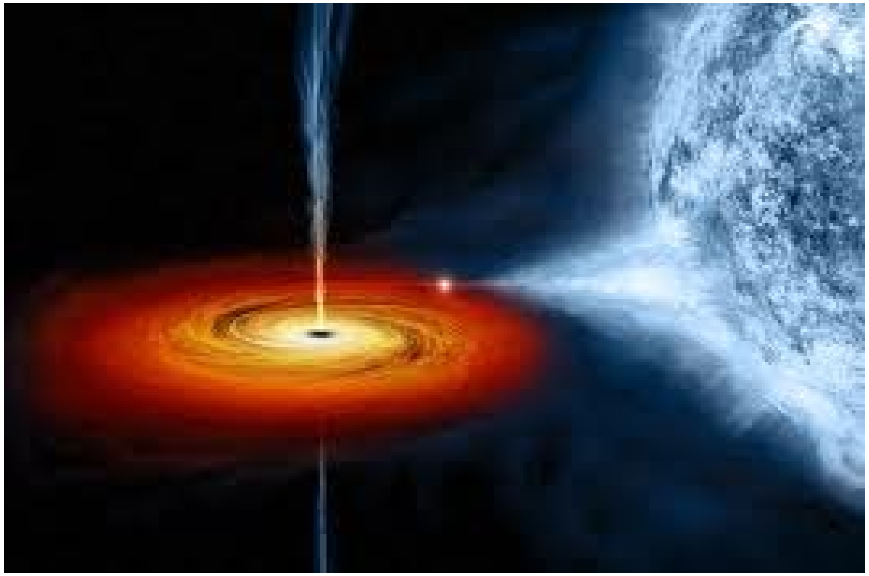
If the photon **in step 2** is still in the Thomson limit (after gaining a factor gamma in energy in **step 1**), then it will gain another factor gamma in energy in **step 3**.

For example, say that gamma = 10 (i.e., electron velocity is ~0.95 c).
Initial photon energy is 0.003 keV (visible light)
After one scattering (**step 1**) the photon has energy of the order 0.03 keV. Since 0.03 keV << 511 keV we are still in the Thomson limit (**step 2**).
We boost the photon again (**step 3**) and the final energy will then be 0.3 keV (soft X-rays)

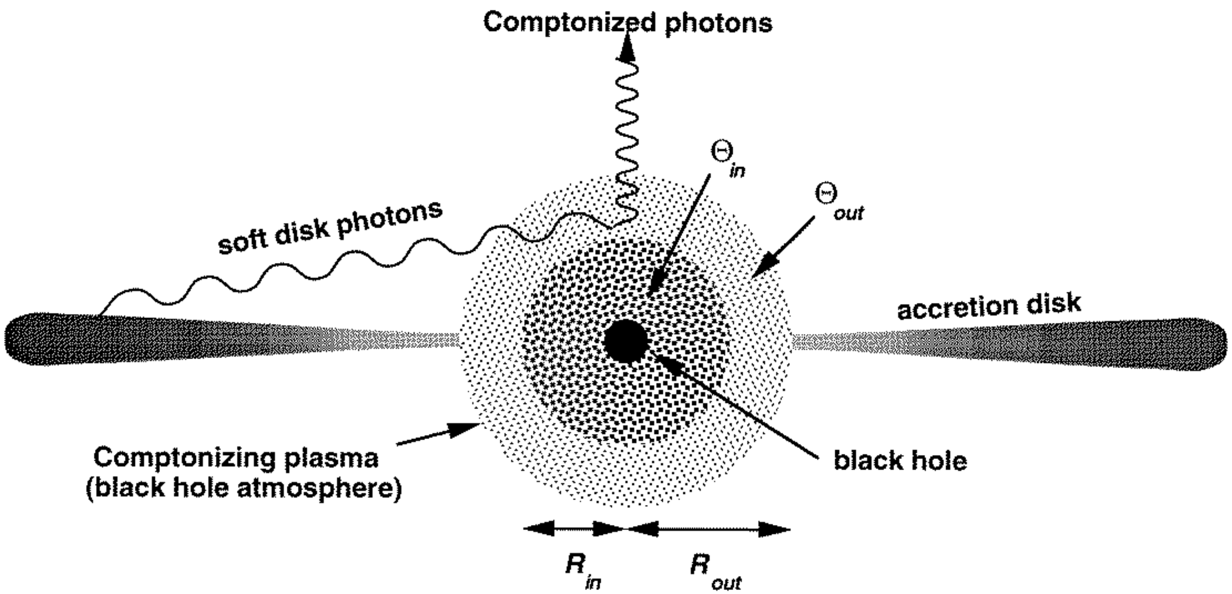



Accreting Neutron Stars
Accreting Black Holes



# Cygnus X-1




15 Msun black hole around a blue supergiant star.

Easily visible in optical with a small telescope (v ~ 9 mag)

# A final step to Inverse Compton

The last equation we derived $\epsilon_1 = \epsilon_1' \gamma (1 + \beta \cos \theta_1')$ might bother you because not all quantities are calculated in the lab frame K.

We can add a final step by transforming the angles from K' to K by using the formula for the aberration of angles.

In this way after some algebra we arrive at the expression:

$$\boldsymbol{\epsilon_1 = \epsilon \frac{1 - \beta \cos \theta}{1 - \beta \cos \theta_1}}$$

# Maximum and Minimum Energy

We can now see that wen theta = pi and theta1 = 0, then the photon is scattered along the electron velocity vector (head-on) and epsilon1 is maximized.

$$\epsilon_1 = \epsilon \frac{1+\beta}{1-\beta}$$

When theta = 0 and theta1 = pi, then the photon scatters “from behind” the electron (tail-on) and epsilon1 is minimized

$$\epsilon_1 = \epsilon \frac{1-\beta}{1+\beta}$$

So, in fact, the photon is *gaining energy* in the former case and *losing energy* in the latter.

Question: what happens if we have many photons coming from all directions? Will inverse Compton produce a higher energy photon distribution or not?

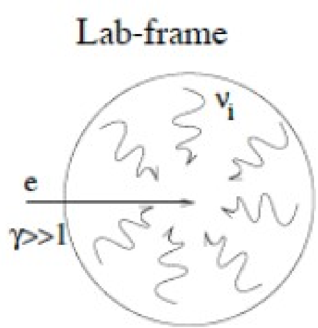
# Answer

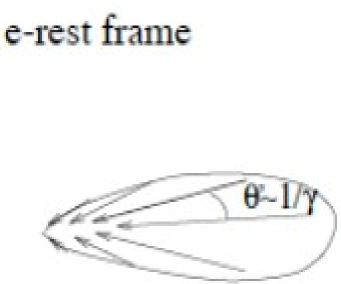
If we have many photons, collisions in which the photon overtakes the electron (in K) will result in a reduced photon energy, but, on average, the energy gain will be positive because there are relatively fewer overtaking than head-on collisions.

# Answer

If we have many photons, collisions in which the photon overtakes the electron (in K) will result in a reduced photon energy, but, on average, the energy gain will be positive because there are relatively fewer overtaking than head-on collisions.

Why?




We'll show later that the average photon energy after one IC scattering is:

$$\langle \epsilon_1 \rangle = \frac{4}{3} \gamma^2 \epsilon$$

# IMPORTANT

This expression refers only to the power in the scattered radiation. It is NOT the power in the radiation *before the scattering.*

*Therefore the **net power** emitted by the electron is:*

[Total energy loss of the electron] =
[Power in radiation after scattering] – [Power in radiation before scattering]

$$\boxed{\frac{dE_1}{dt} = c\,\sigma_T\,\gamma^2\left(1+\frac{1}{3}\beta^2\right)U_{ph}}$$

$$\boxed{\frac{dE_1}{dt} = -\sigma_T\,c\,U_{ph}}$$

$$P_{compt} = \frac{dE_{rad}}{dt} = c\,\sigma_T\,\gamma^2\left(1+\frac{1}{3}\beta^2\right)U_{ph} - \sigma_T\,c\,U_{ph} = \frac{4}{3}\sigma_T\,c\,\gamma^2\beta^2\,U_{ph}$$

# Synchrotron vs. Compton Power

$$P_{compt} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{ph}$$

$$P_{synch} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_B$$

Why are these two powers so similar?

After all we have a very different physical mechanism operating.

# Synchrotron vs. Compton Power

$$P_{compt} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{ph}$$

$$P_{synch} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_B$$

The reason for this is that the energy loss rate depends upon the electric field which accelerates the electron in its rest frame and it does not matter what the origin of that field is.

In the case of synchrotron radiation, the electric field is the electric field (**E = v** × **B**) due to motion of the electron through the magnetic field whereas, in the case of inverse Compton scattering, it is the sum of the electric fields of the electromagnetic waves incident upon the electron.

Some photons (tail-on) will be *downscattered*
Some (most) photons (head-on) will be *upscattered*

Therefore the low frequency part of the spectrum will be filled by downscattered photons. The high frequency part by upscattered photons

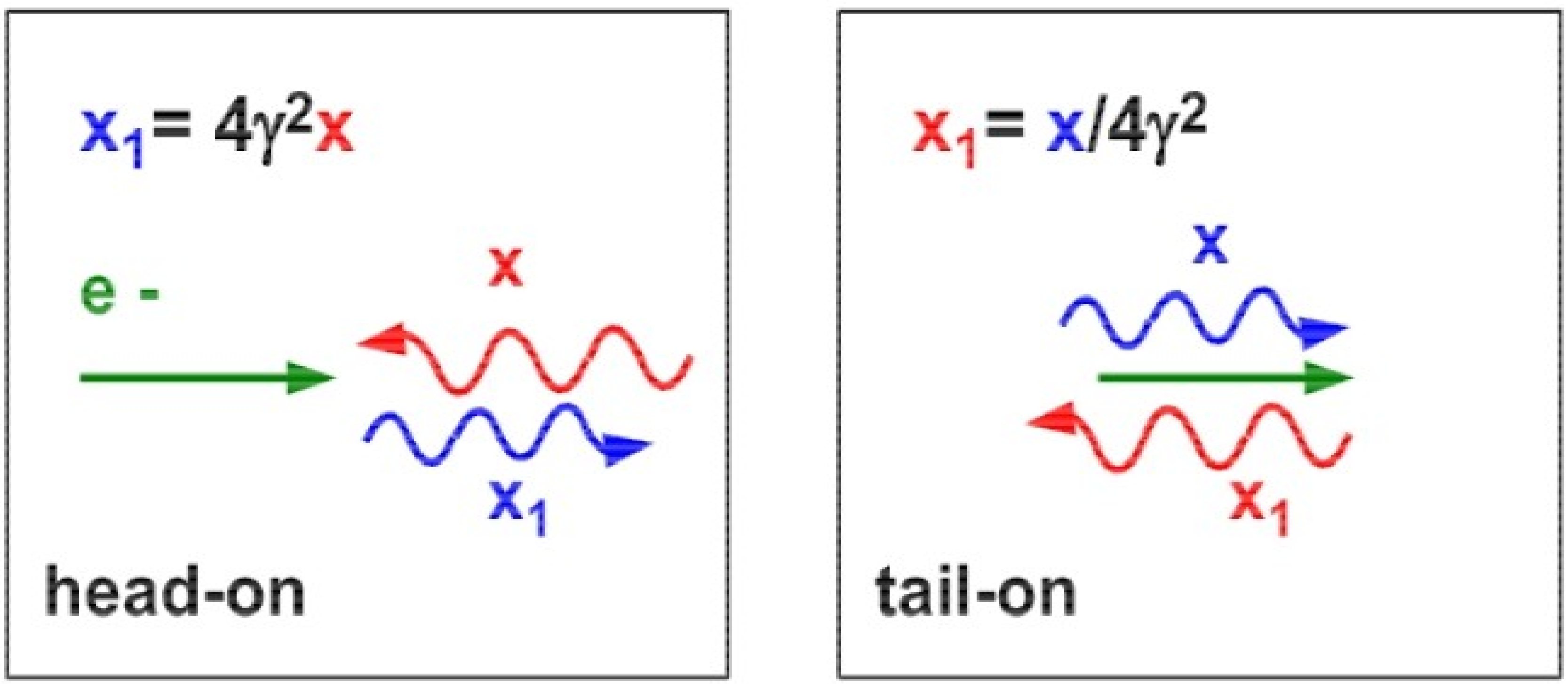
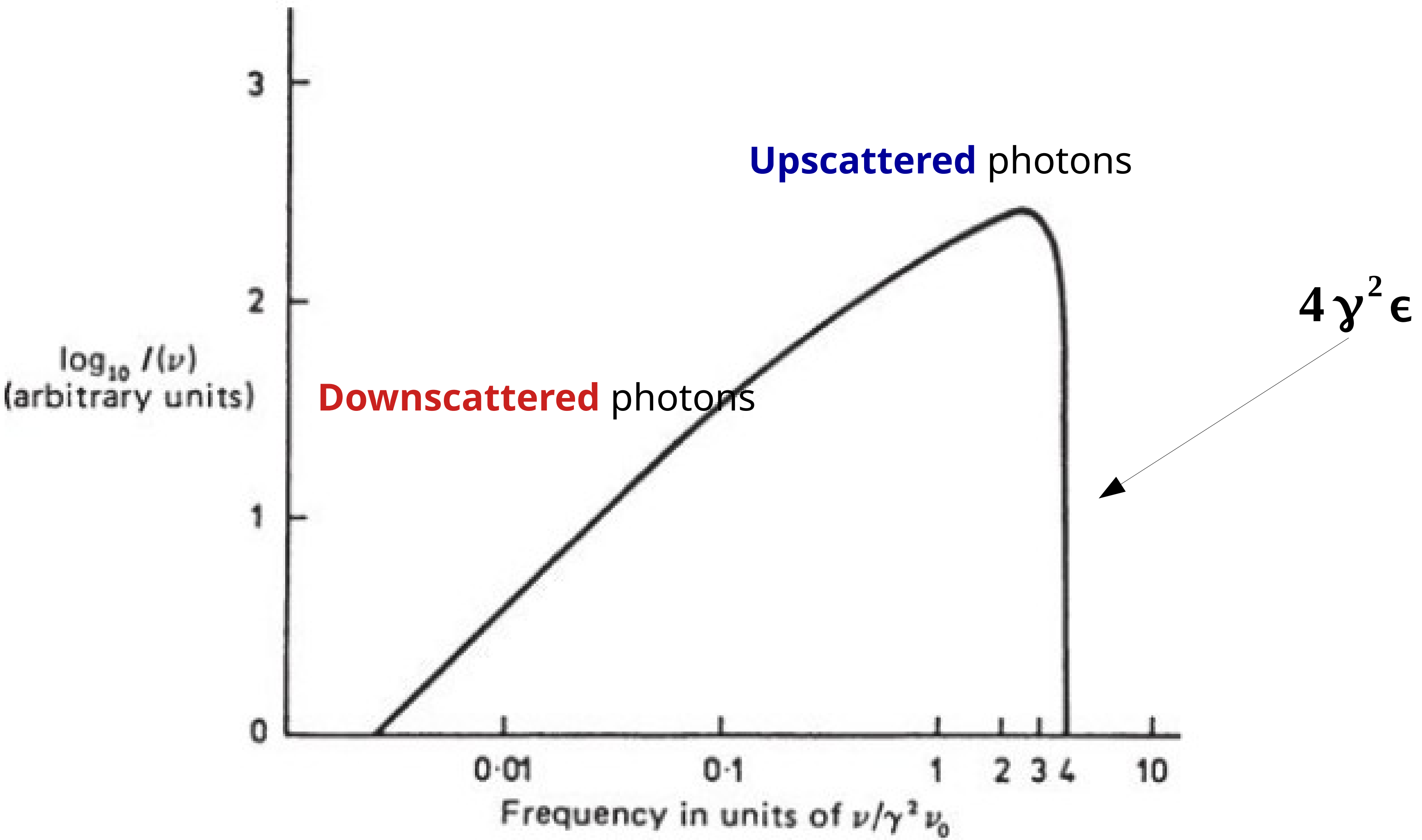



Image from the book of Ghisellini

# Single Particle Spectrum



# Multiple Scatterings $\gamma h\nu \ll mc^2$

$$y = \text{[average \# of scatt.]} \times \text{[average fractional energy gain for scatt.]}$$

**Comptonization parameter**

**Important: If y>1 then Comptonization is important.
(This does not necessarily mean
that there are many scatterings...)**

# Multiple Scatterings $\gamma h\nu \ll mc^2$

$$y = [\text{average \# of scatt.}] \times \boxed{[\text{average fractional energy gain for scatt.}]}$$

**Let's first evaluate the average fractional energy change per scattering**

# Average energy

We have seen that when (monochromatic) photons scatter (once) off **one** relativistic electron, then:

$$\langle \epsilon_1 \rangle = \frac{4}{3} \gamma^2 \epsilon$$

Gamma depends on the energy of the electron and has a single value since there is only one electron. So the question now is what is <gamma^2> ? In other words, if we have many (relativistic) electrons, what happens to the average gain in energy after one scattering?

We will start first by assuming *a thermal distribution of electrons.* The reason for this is that this case is the simplest and will help us to find a few general parameters.

# Thermal Compton (**ultra-rel. case**)

We have the following starting assumptions:

1. thermal distribution of electrons
2. electrons are relativistic
3. $\gamma h\nu \ll mc^2$

The relativistic Maxwell-Boltzmann distribution has the form:

$$N(\gamma)=\gamma^2 e^{-\gamma/\Sigma}$$

$$\Sigma=\frac{kT}{mc^2}$$

We want to find $\langle\gamma^2\rangle$ and so we can use the MB distribution to calculate this average.

# Thermal Compton (**ultra-rel. case**)

$$\langle \gamma^2 \rangle = \frac{\int \gamma^2 \gamma^2 e^{-\gamma/\Sigma} d\gamma}{\int \gamma^2 e^{-\gamma/\Sigma} d\gamma} = 12\,\Sigma^2 = 12 \left( \frac{kT}{mc^2} \right)^2$$

Therefore we now know that when there is a thermal distribution of ultra-relativistic electrons, the average gain in energy of the photons per scattering is proportional **to the square of the thermal energy** of the electrons

$$y = [\text{average \# of scatt.}] \times \boxed{[\text{average fractional energy gain for scatt.}]}$$

# Thermal Compton (**ultra-rel. case**)

Let's now calculate the average number of scatterings

$$y = \boxed{[\text{average } \# \text{ of scatt.}]} \times [\text{average fractional energy gain for scatt.}]$$

To calculate the average number of scattering you should think at the path that the photon does in the cloud of electrons as a “random walk”.

# Multiple Scatterings ($\gamma h\nu \ll mc^2$)

$$y = \boxed{[\text{average \# of scatt.}]} \times [\text{average fractional energy gain for scatt.}]$$

| **Case 1: tau>1** | **Case 2: tau<1** |
|---|---|
| Average # of scatt. = $\tau_T^2$ | Average # of scatt. = $\tau_T$ |

# Thermal Compton (**non rel. case**)

We have everything we need now for calculating the spectrum for photons doing multiple scatterings off a thermal distribution of relativistic electrons (when we are in the Thomson limit).

Before continuing though, let's first check what happens when the electrons are non relativistic. The average # of scatt. does not change wrt the previous case. We need to calculate only the change in fractional energy.

$$y = [\text{average \# of scatt.}] \times \boxed{[\text{average fractional energy gain for scatt.}]}$$

NOTE: Non-relativistic here means that $kT \ll mc^2$

# Compton Spectrum (ultra-rel. and tau<1)

$$x \equiv \frac{h\nu}{k_B T}$$

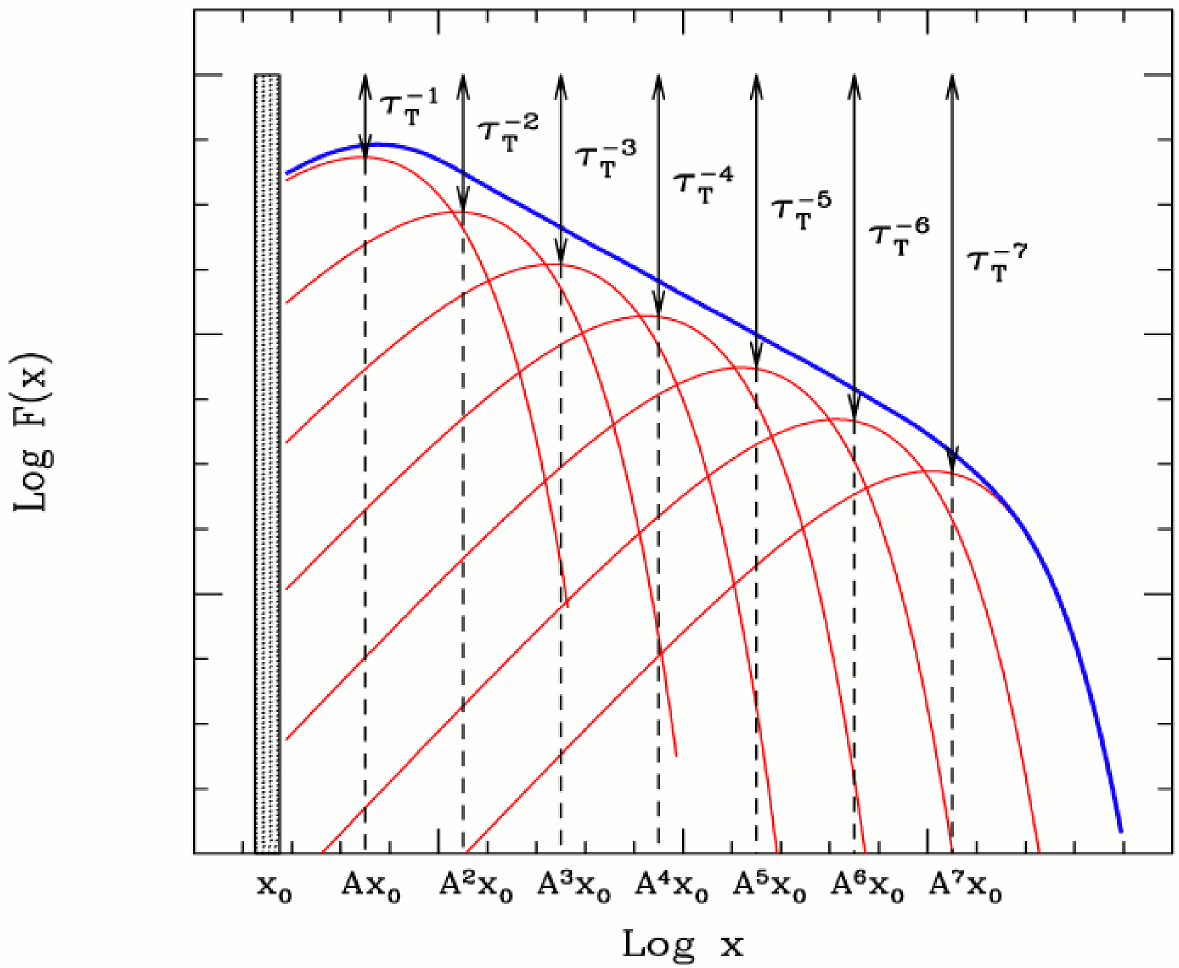
Now, if the medium is of small optical depth the probability of a photon undergoing k scatterings before escaping the Comptonizing cloud is approximately:

$$Prob. \approx \tau^k$$

The intensity of the emerging Compton radiation will therefore decrease by a factor tau as a function of energy (or frequency) of the radiation.

The spectrum will look like in the figure to the right.

Note that F0 = F1/tau = F2/tau^2 etc...



# Compton scattering (non rel.)

When the photons diffuse along the energy axis (i.e. sometimes they lose, but more often they gain energy), then the time evolution is described by a diffusion equation. Most easily this equation is written in terms of the phase-space density (occupation number) n(x) of photons of energy x = epsilon/kT.

The diffusion equation is called in this case the **Kompaneet equation:**

$$\frac{\partial n}{\partial y} = \left(\frac{1}{x^2}\right) \cdot \frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right] \qquad x \equiv \frac{h\nu}{k_B T}$$

Generally, the solutions of the equation **have to be found numerically**, but there are a number of cases in which analytic solutions can be found.

# Compton scattering (non rel.)

$$\frac{\Delta\epsilon}{\epsilon} = \frac{(4kT-\epsilon)}{mc^2}$$

Recoil effect (Downscattering & Cooling)

Upscattering by thermal electrons

Non-linear term (induced scattering)

$$\frac{\partial n}{\partial y} = \left(\frac{1}{x^2}\right)\cdot\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x}+n+n^2\right)\right]$$

$$x \equiv \frac{h\nu}{k_B T}$$

Generally, the solutions of the equation **have to be found numerically**, but there are a number of cases in which analytic solutions can be found.

# Solutions to the Kompaneet Equation

A family of (analytical) solutions for the Kompaneet Equation can be found when the amount of Comptonization is either weak (y<<1), relatively strong (y~1) or very strong (y>>1).

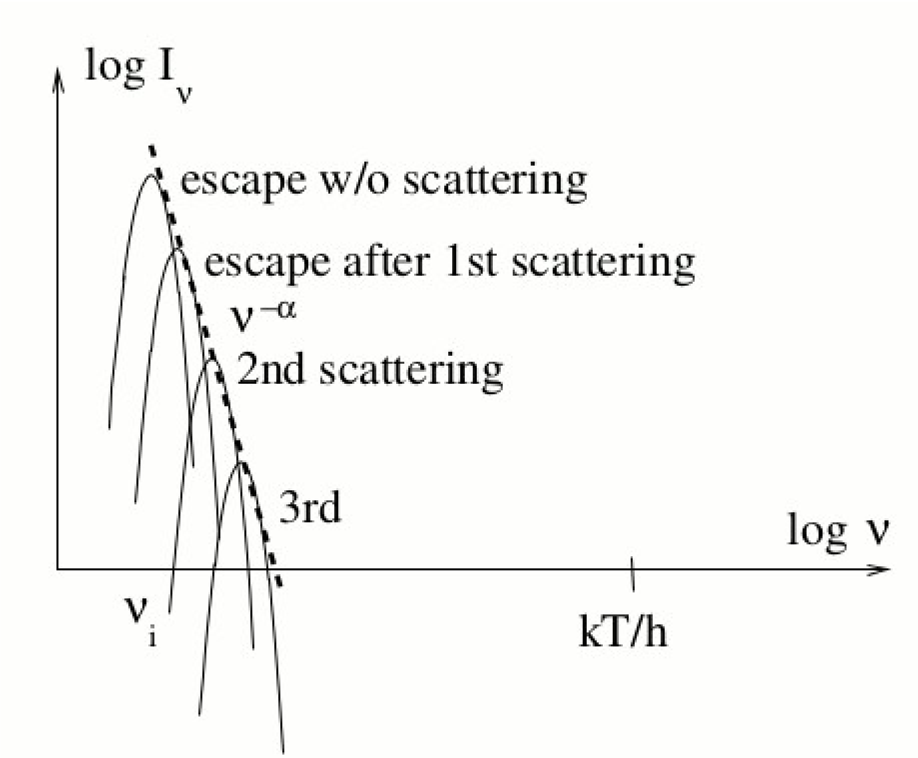
In the first two cases the solution is a power-law spectrum with spectral index alpha:

$$\alpha = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y}}$$

The positive root is a solution when y~1
The negative root is a solution when y<<1

# Case y<<1: Very unsaturated

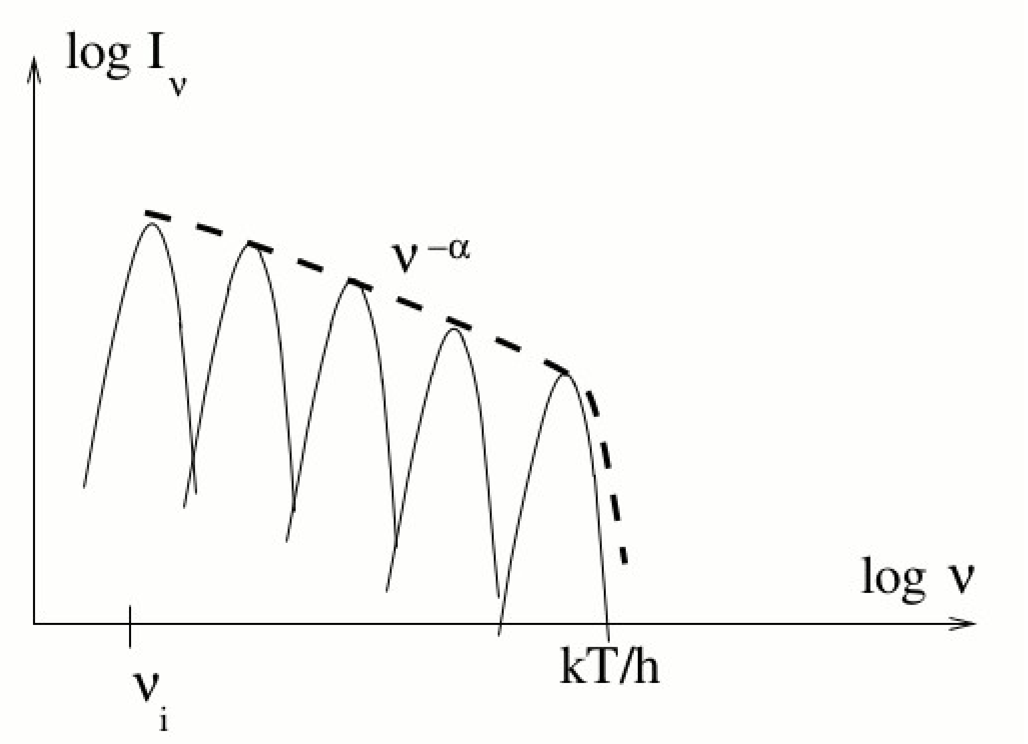
In this case alpha~2/sqrt(y)>>1




Only a small fraction of photons interact and are Comptonized. A steep power-law is created

# Case y~1: unsaturated

In this case alpha~1



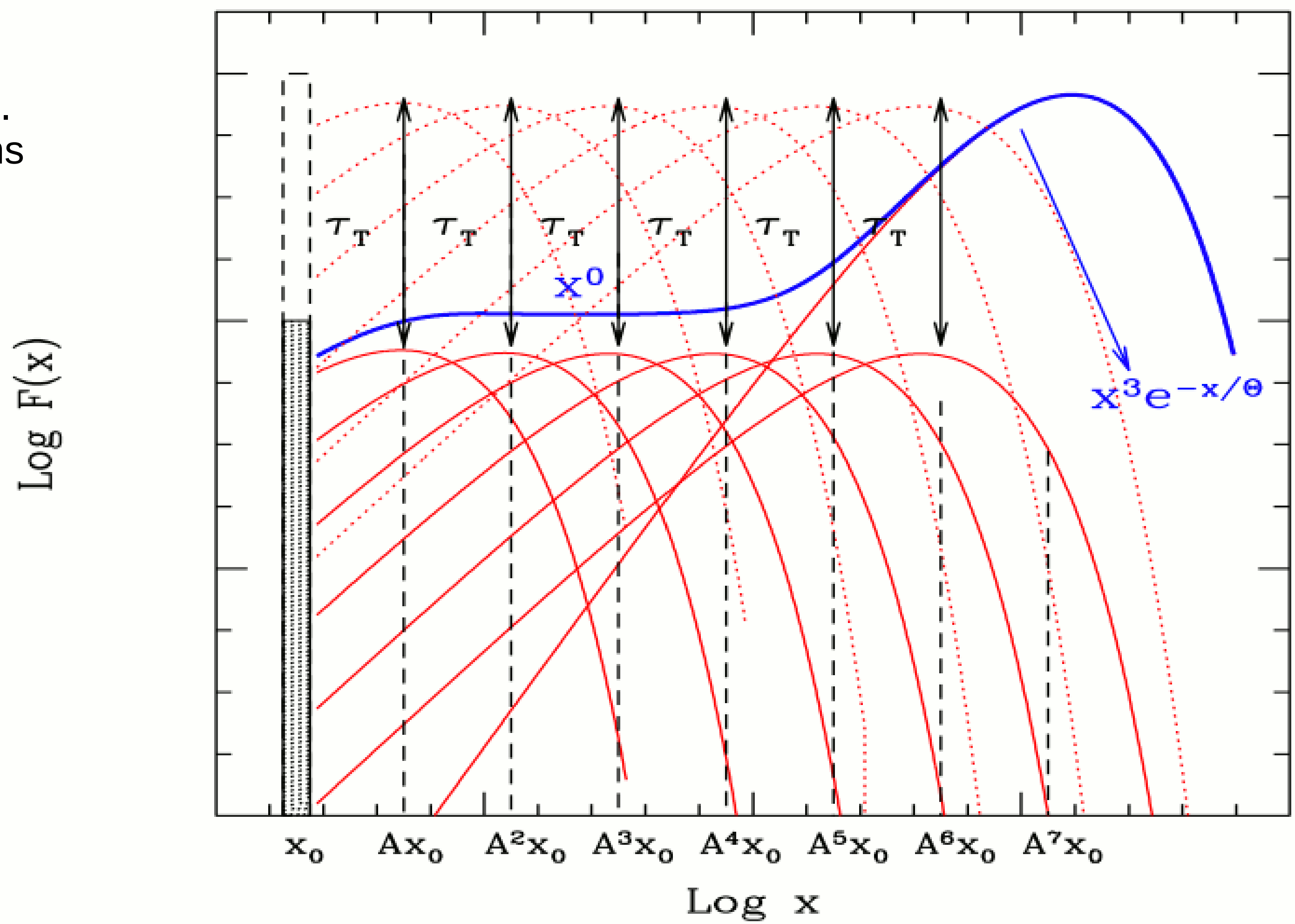

Examples: hard X-ray spectra
of Galactic black hole candidates,
accreting neutron stars,
some active galactic nuclei (Seyferts).

# Case y>>1: Saturation

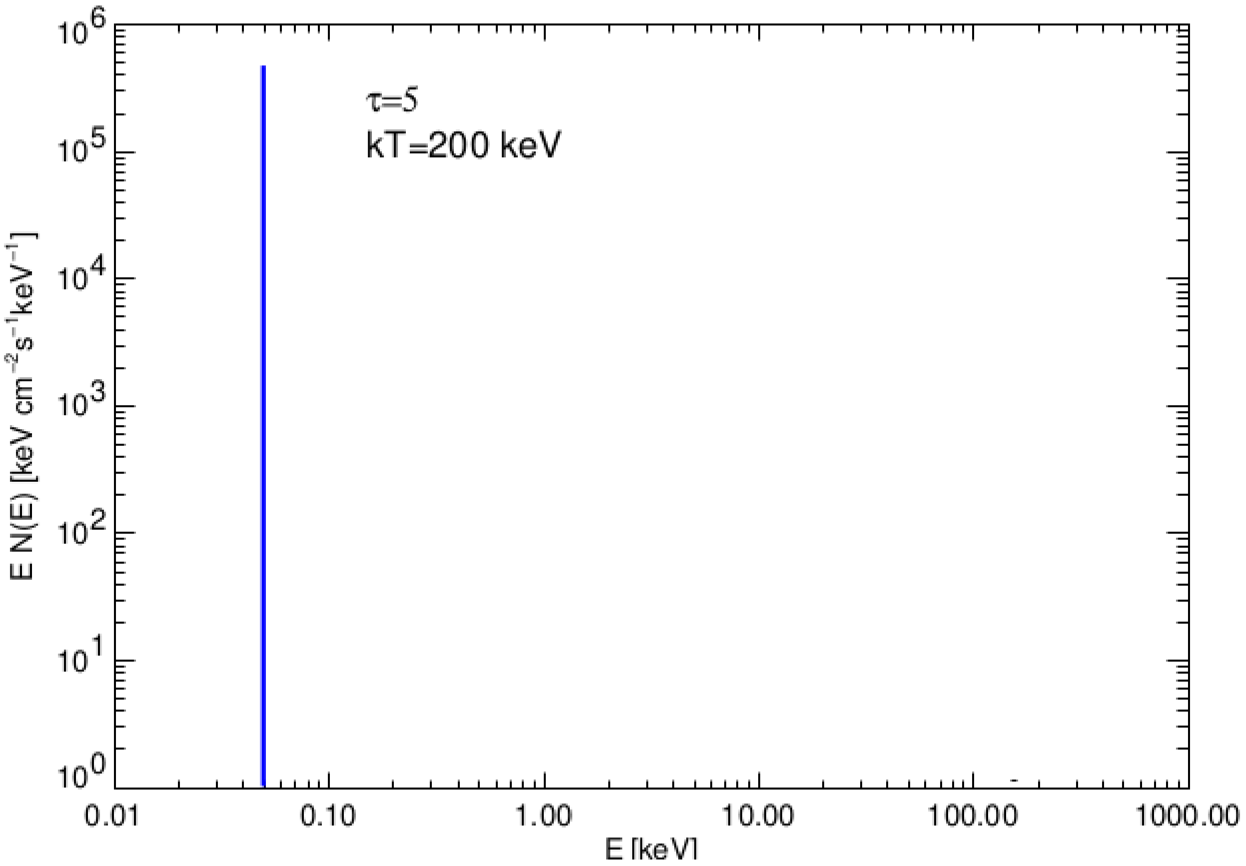
For the first scattering order, nearly all photons are scattered. Therefore the number of photons escaping at each scattering order is the same.

However, this cannot be going on indefinitely, since there is a limit from the energy of the electrons.

When this limit is reached, the photons show a “bump” which has the same frequency dependency of the Wien spectrum (high energy tail of the blackbody emission).

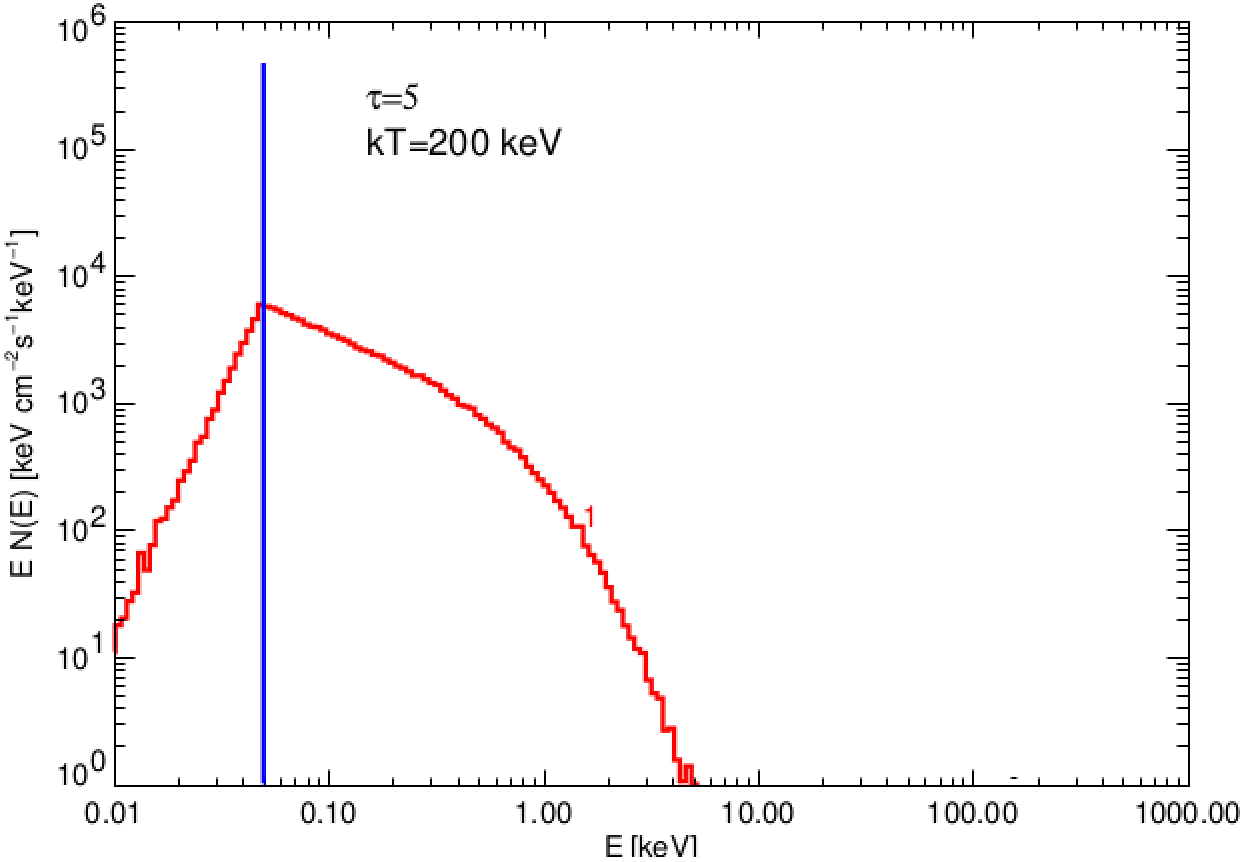


# Example: Numerical Calculations



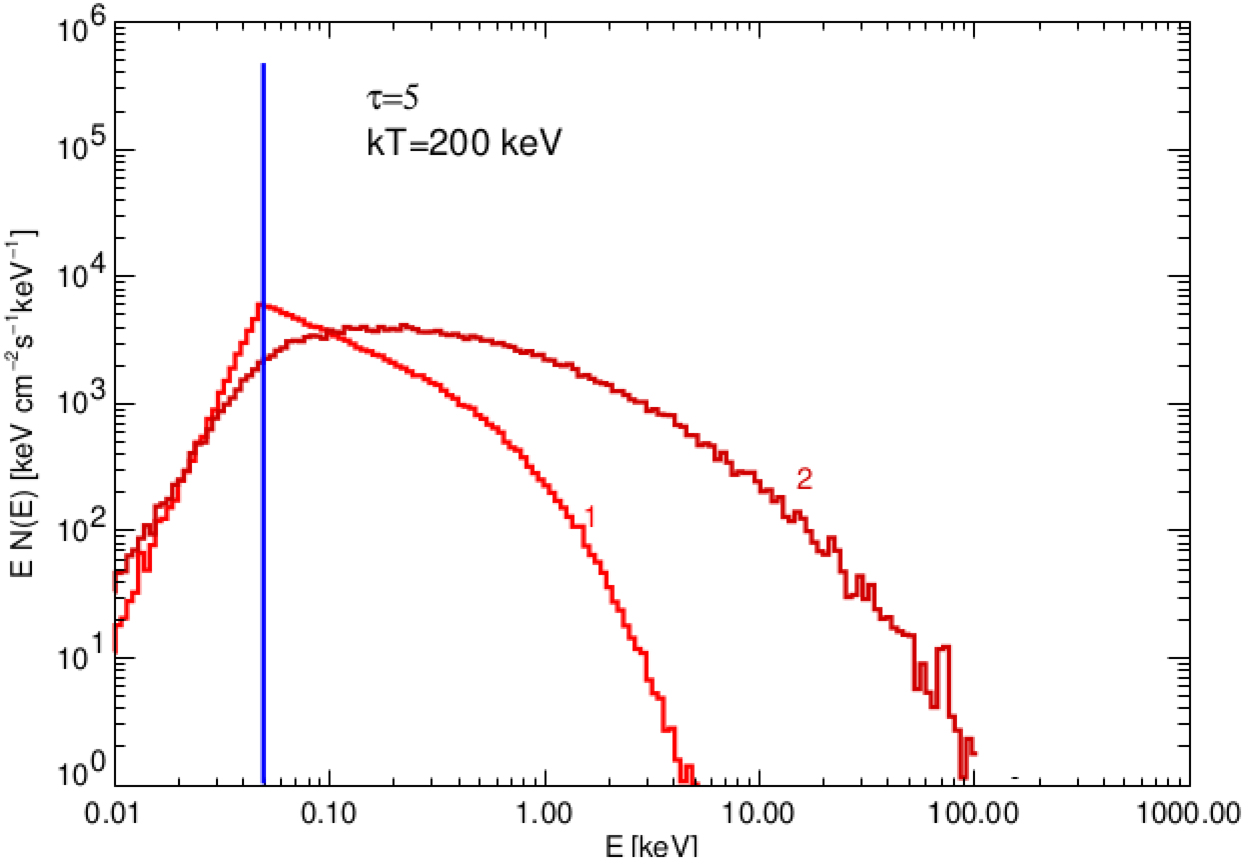

Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

# Example: Numerical Calculations



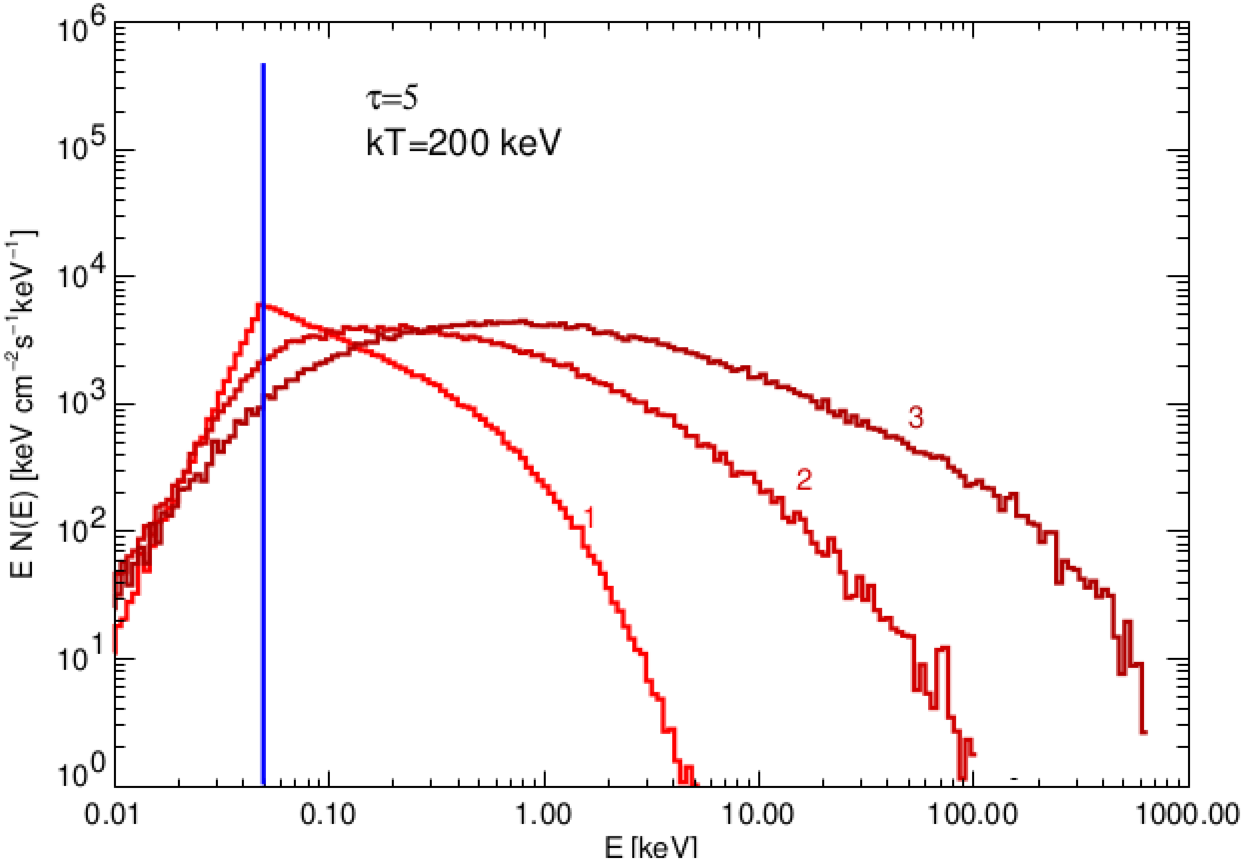

Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

# Example: Numerical Calculations



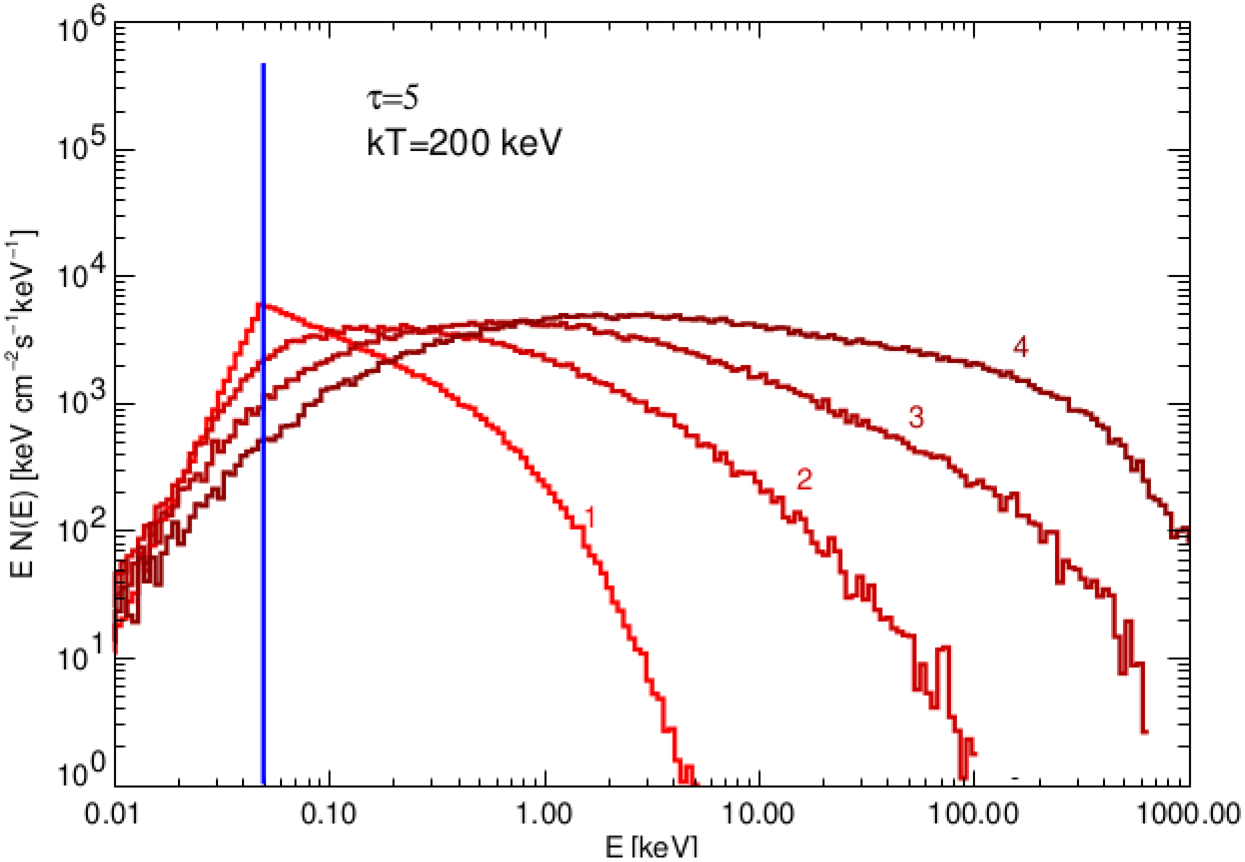

Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

# Example: Numerical Calculations



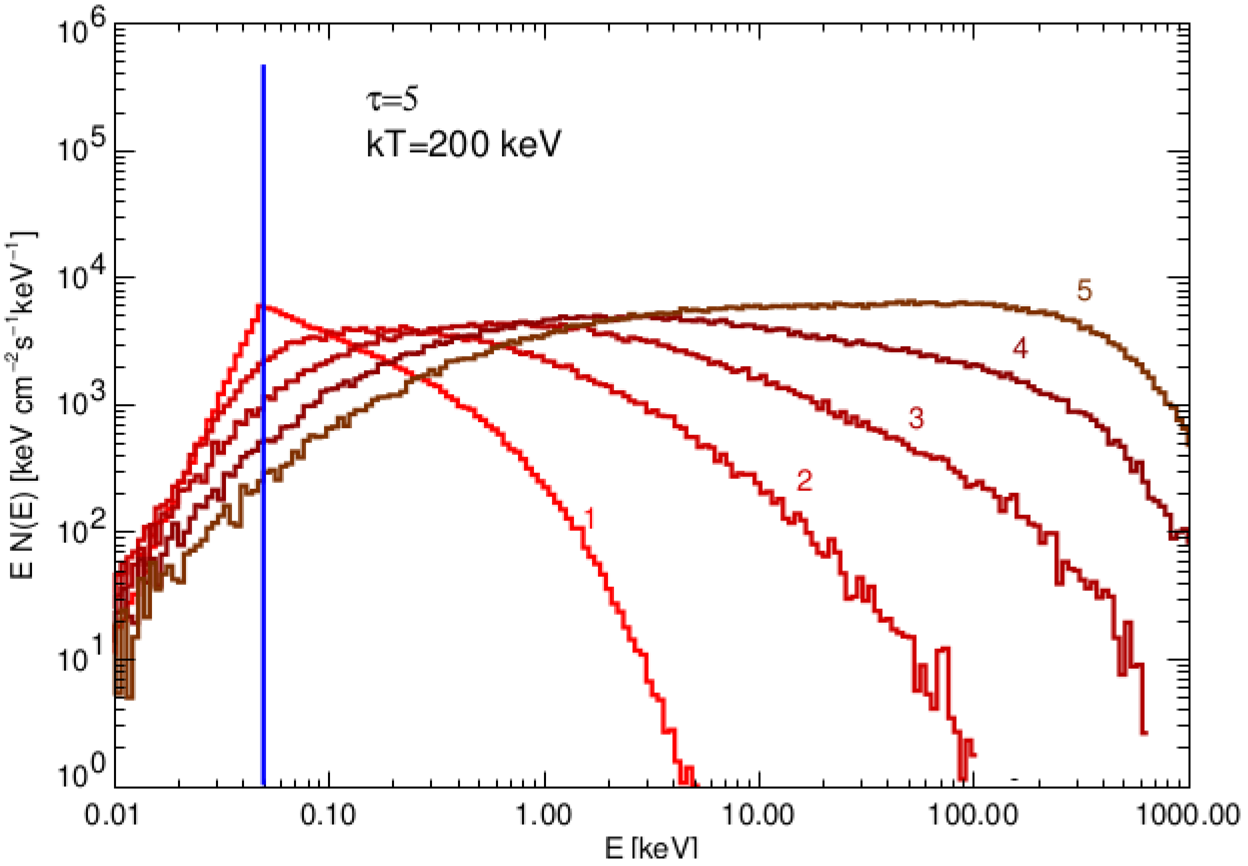

Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

# Example: Numerical Calculations



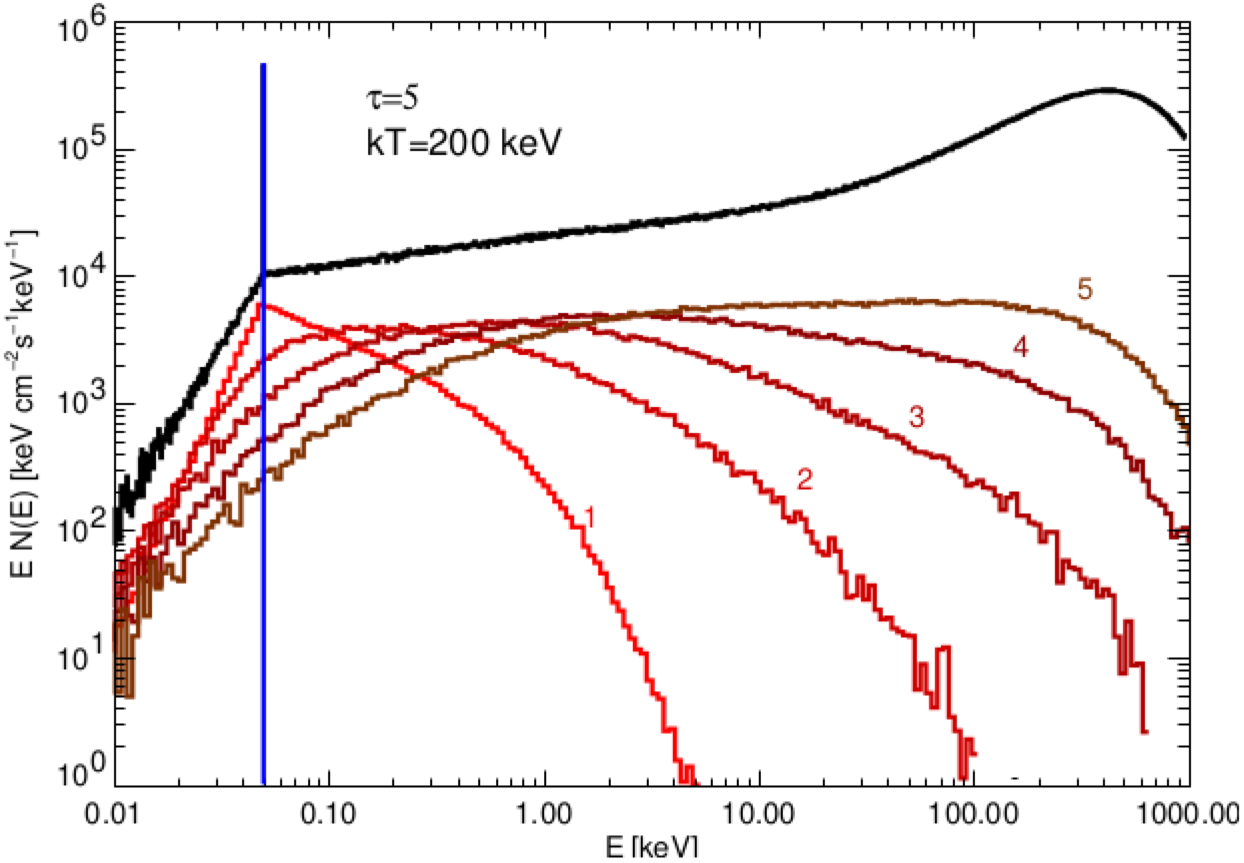

Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

# Example: Numerical Calculations




Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

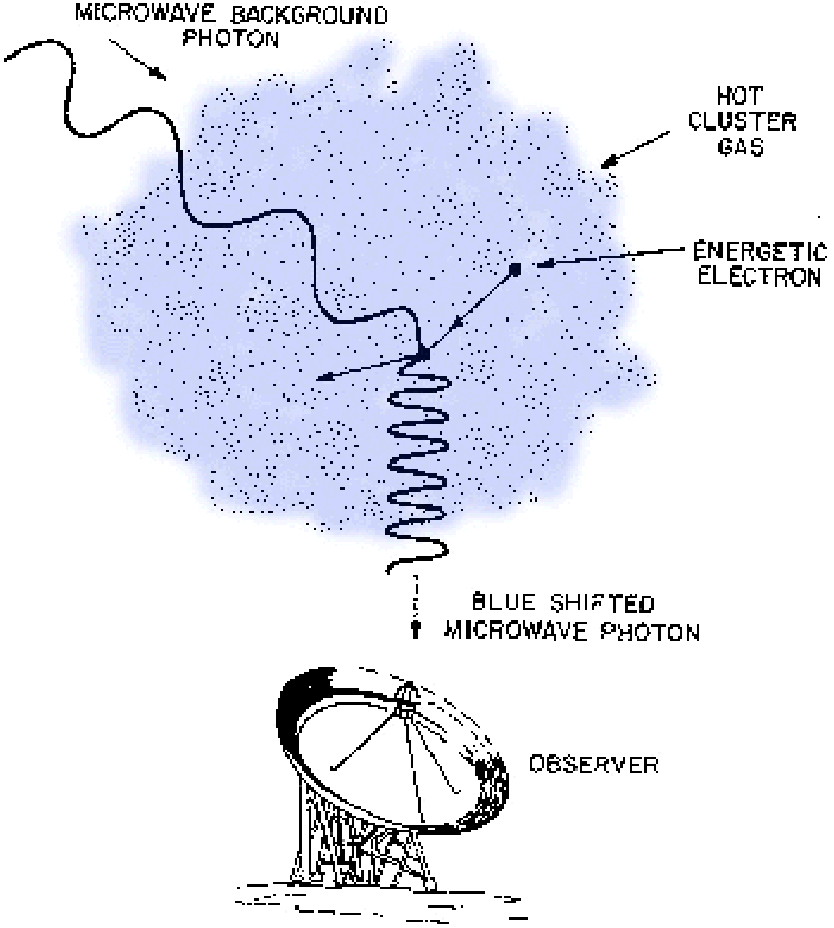
# Example: Numerical Calculations



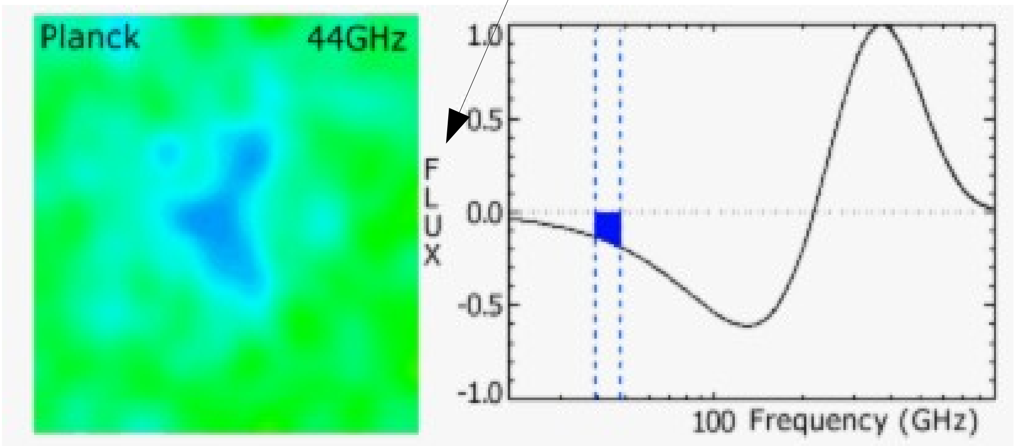

Taken from lecture notes of Joern Wilms (see slides and notes: https://pulsar.sternwarte.uni-erlangen.de/wilms/teach/astrospace/

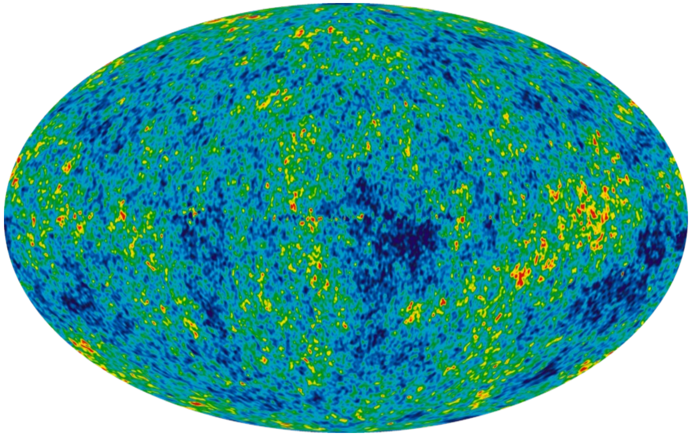
# Case y<<1: Very unsaturated

## Sunyaev Zel'dovich Effect



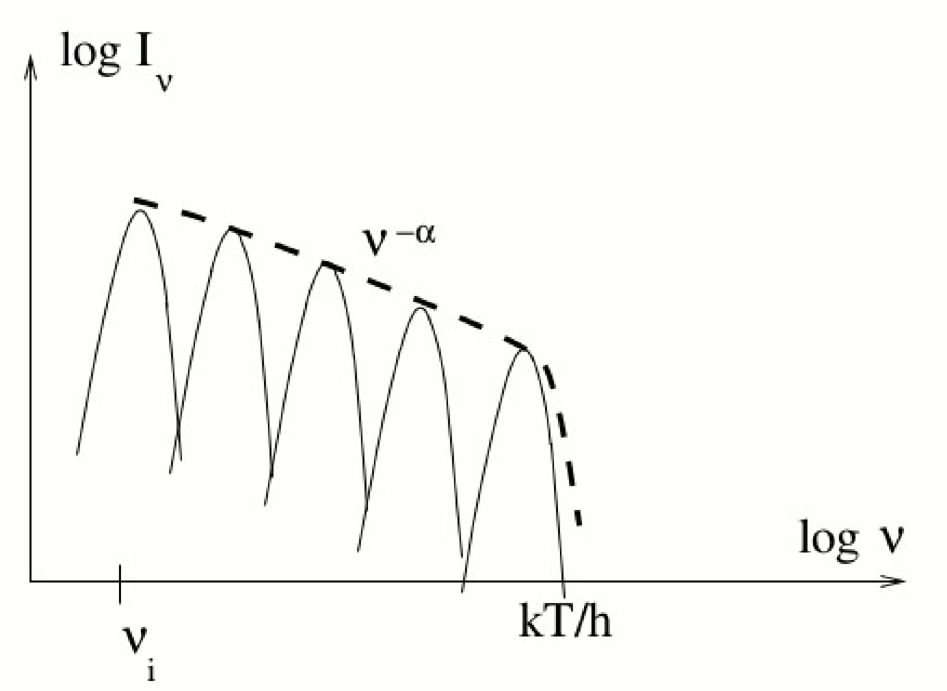

(This is really Delta I_nu)

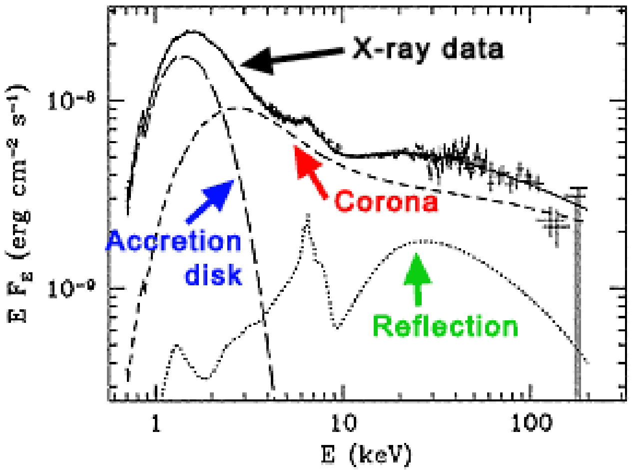


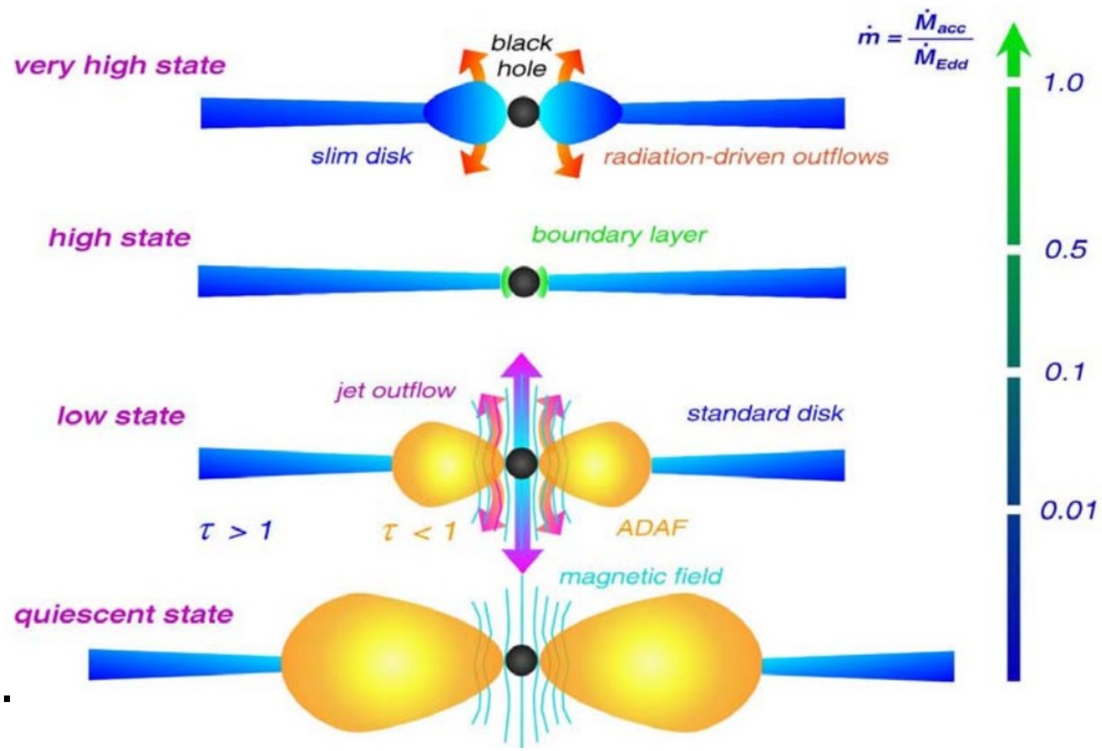



# Case y~1: unsaturated

In this case alpha~1

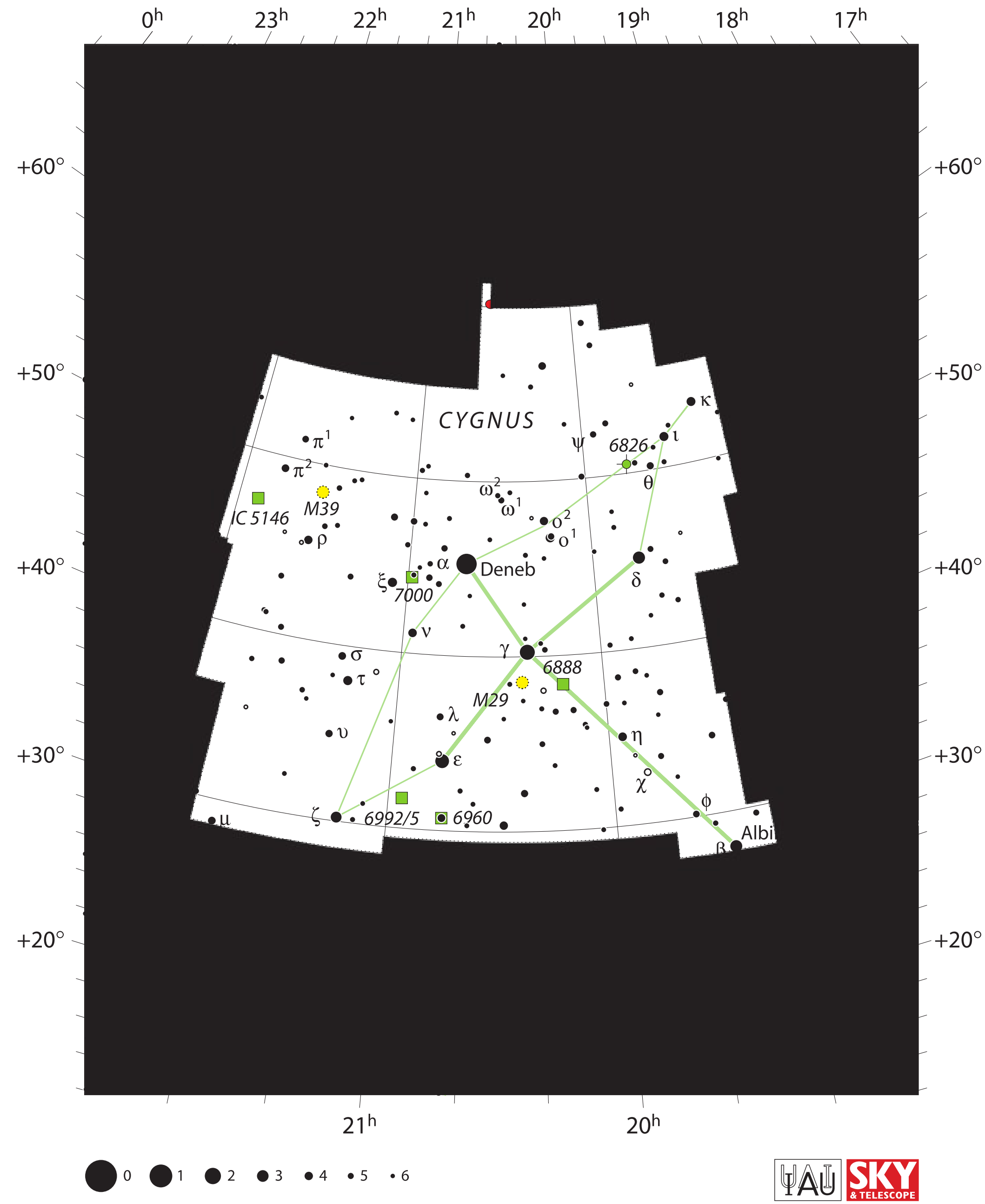


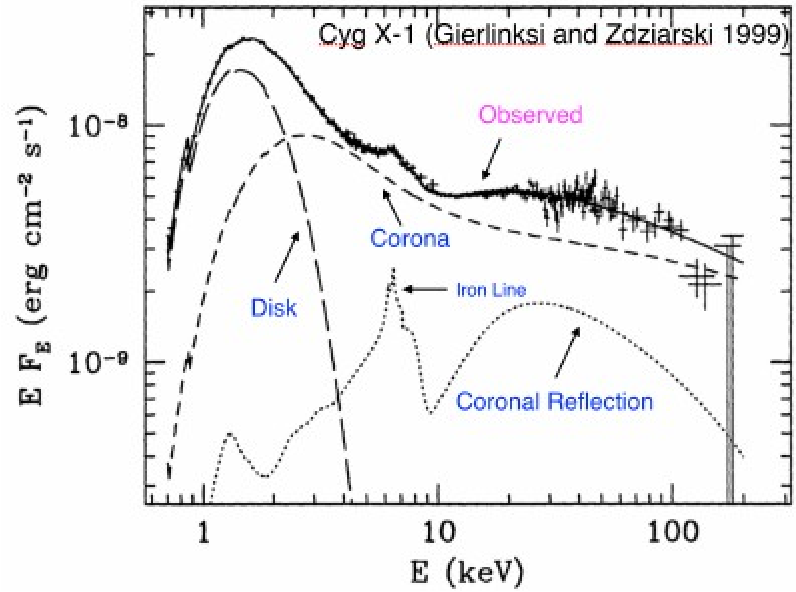




Examples: hard X-ray spectra
of Galactic black hole candidates,
accreting neutron stars,
some active galactic nuclei (Seyfert).

# Comptonization

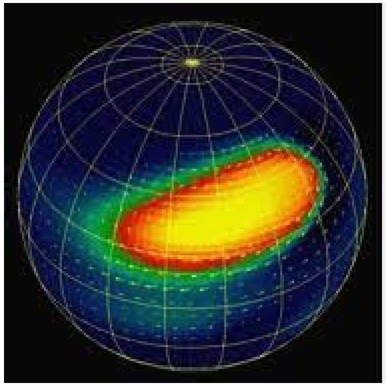


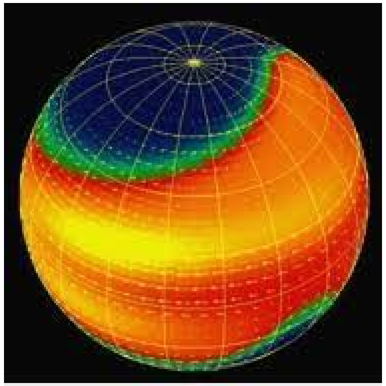



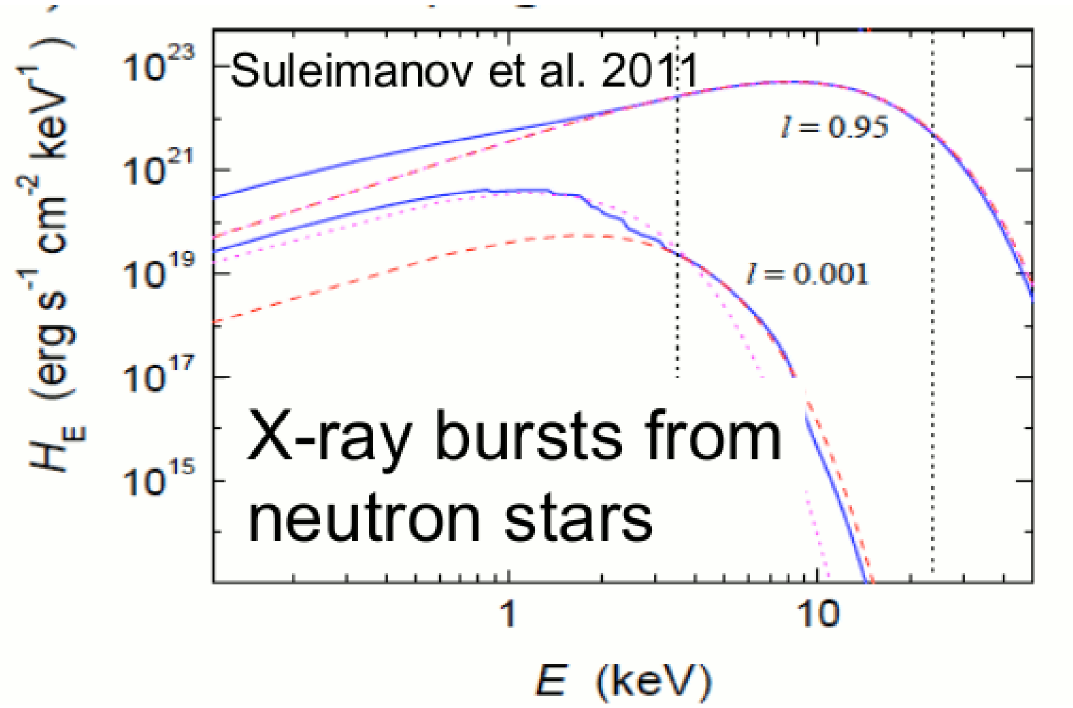


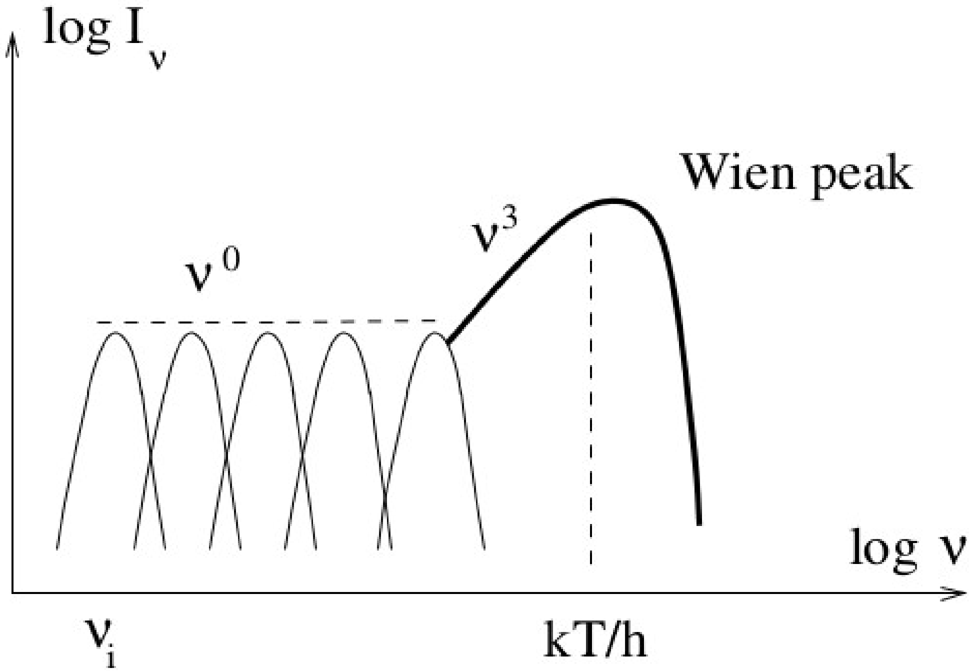
**Apparent magnitude V ~ 9 mag.**
**Easily visible with a small telescope/binocular!**

# Case y>>1: saturated








**Thermonuclear bursts on neutron stars**

# Comptonization: Non-Thermal Distribution

Let's start with relativistic electrons with an isotropic non-thermal distribution:

$$N(\gamma) = K\gamma^{-p} = N(E)\,\frac{dE}{d\gamma}; \quad \gamma_{\min} < \gamma < \gamma_{\max}$$

Then for simplicity, let's take a monochromatic field of photons with frequency $\nu_0$

From our previous discussion, we know the average Compton frequency, and we can find an expression for the Lorentz factor in this way:

$$\nu_c = \frac{4}{3}\gamma^2\nu_0 \rightarrow \gamma = \left(\frac{3\nu_c}{4\nu_0}\right)^{1/2} \rightarrow \left|\frac{d\gamma}{d\nu}\right| = \frac{\nu_c^{-1/2}}{2}\left(\frac{3}{4\nu_o}\right)^{1/2}$$

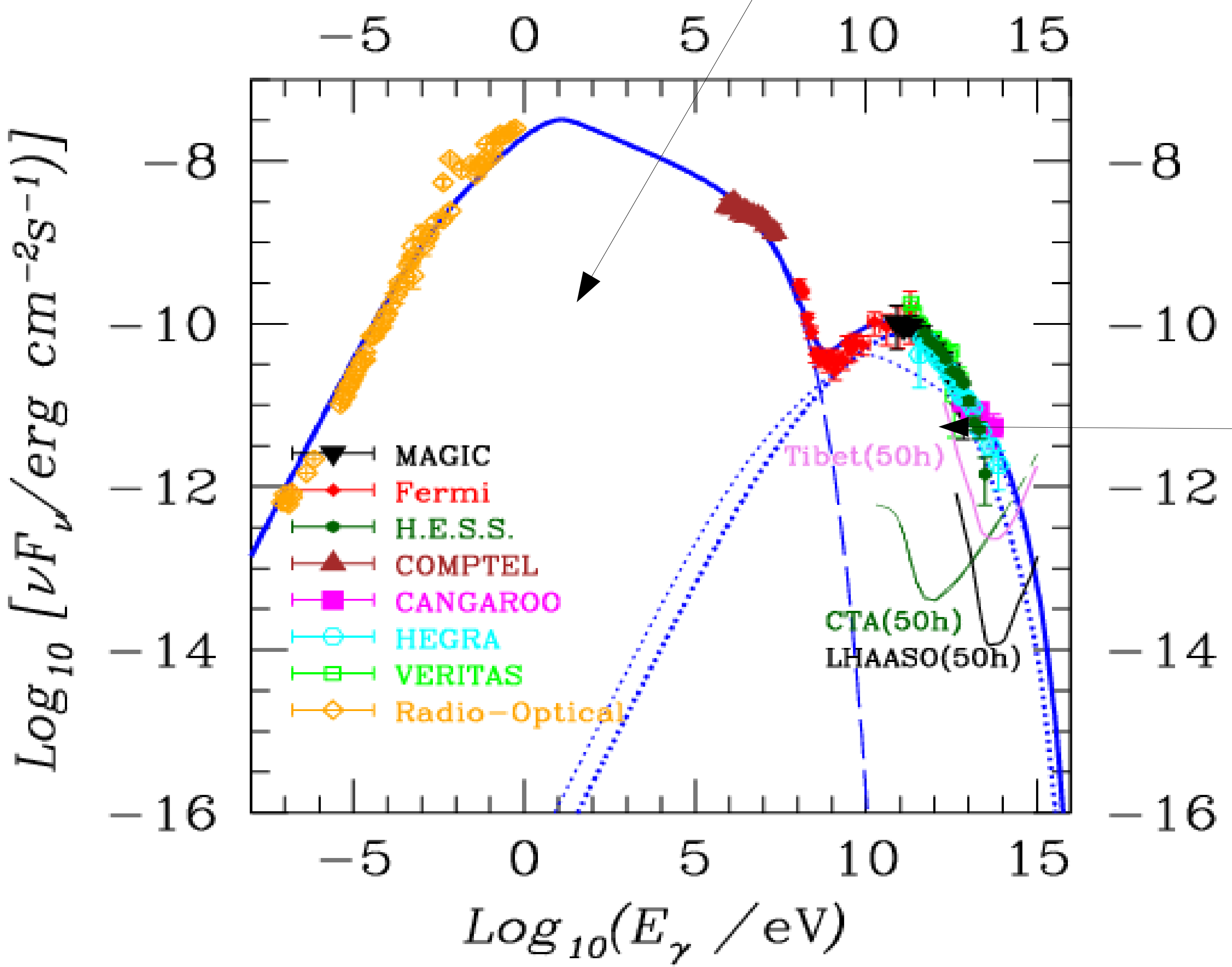
Now we can derive an emission (energy/volume/time/frequency):

$$\varepsilon_\nu\, d\nu = P(\gamma)\, N(\gamma)\, d\gamma \rightarrow \varepsilon_\nu = P(\gamma)\, N(\gamma)\, \frac{d\gamma}{d\nu} = \frac{(4/3)^\alpha}{2}\,\sigma_T c K\,\frac{U_r}{\nu_0}\left(\frac{\nu_c}{\nu_0}\right)^{-\alpha}$$

where $\alpha = \frac{p-1}{2}$

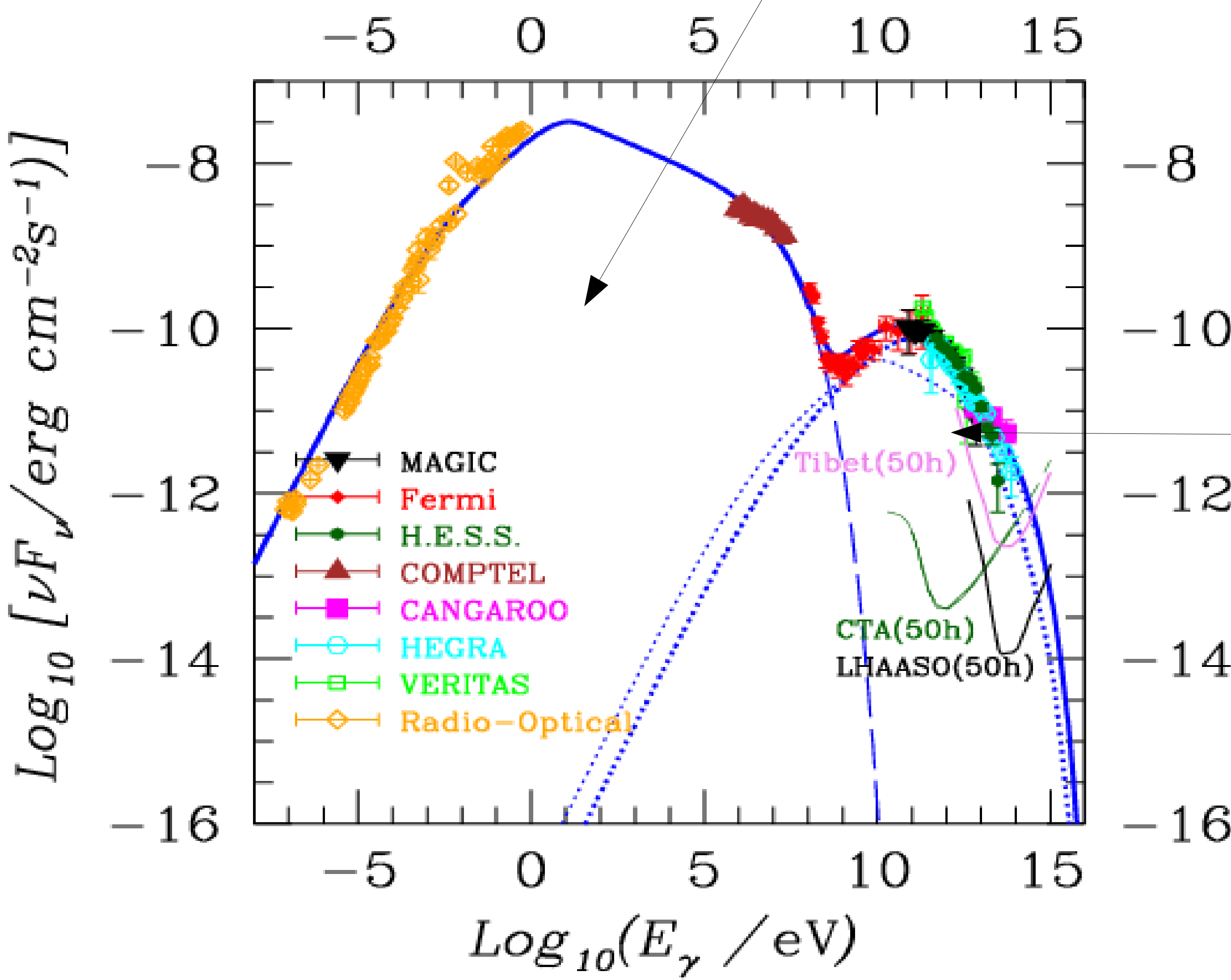
Note again the similarity with synchrotron radiation [power law *p* gives a power law with a slope *alpha = (p-1)/2*]

Synchrotron spectrum of the Crab Nebula.



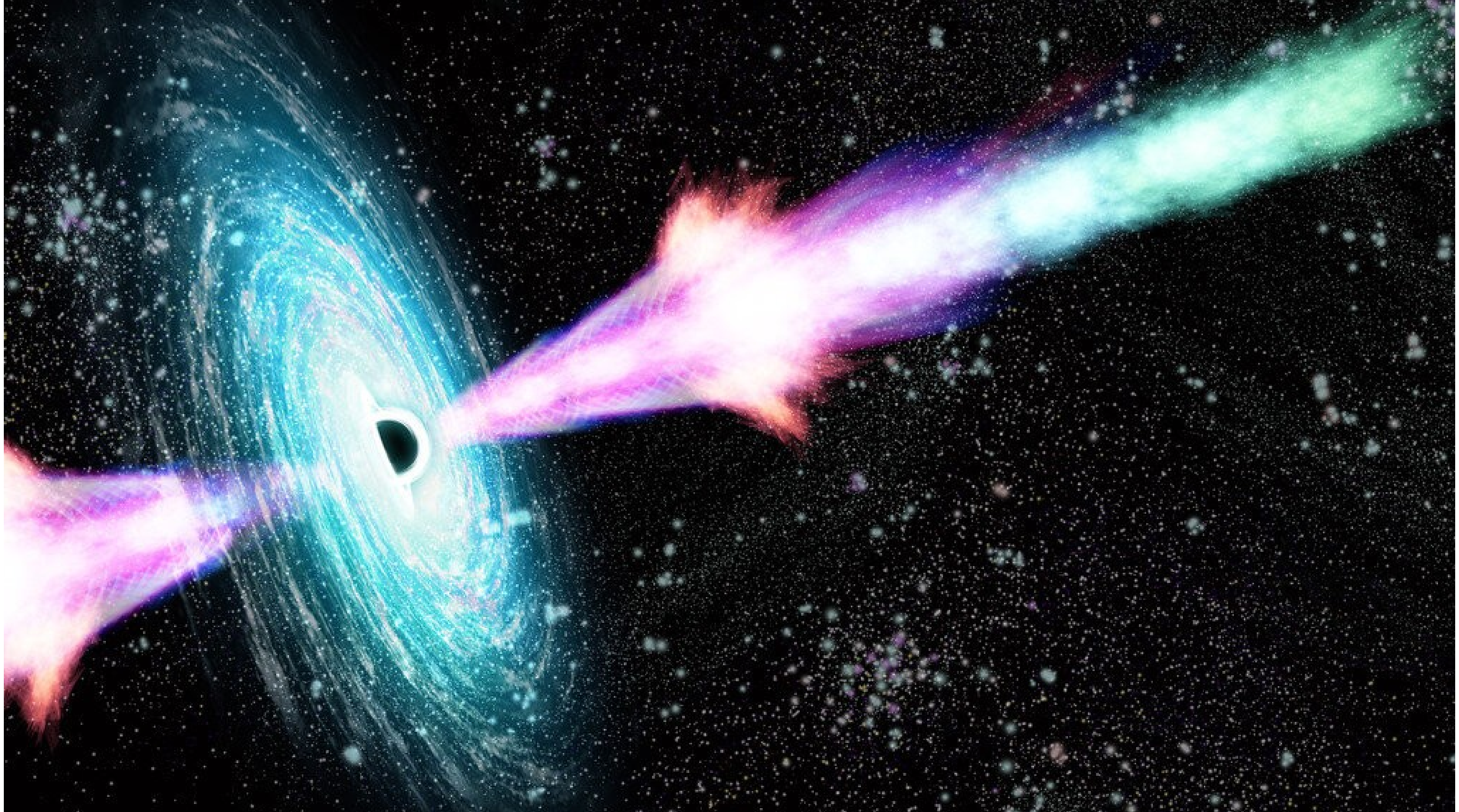

What is this bump here?

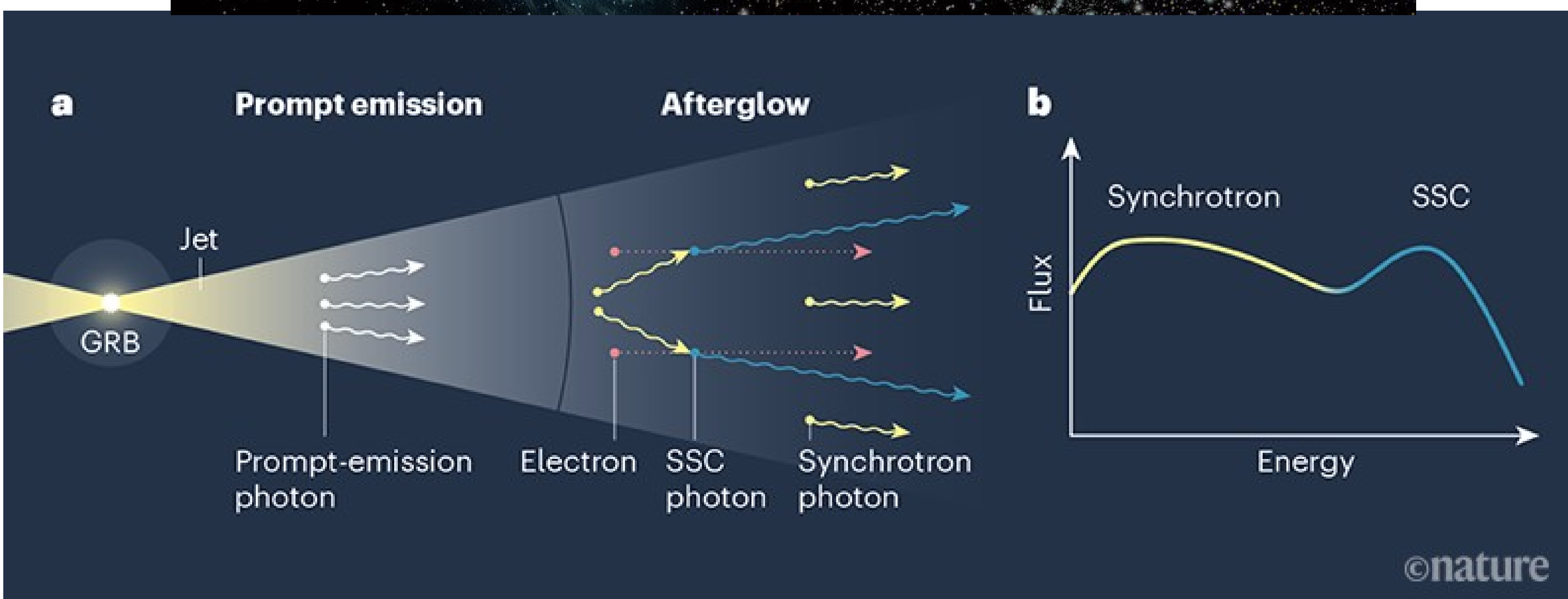
Synchrotron spectrum of the Crab Nebula.




Synchrotron Self Compton

$$\boxed{\frac{P_{compt}}{P_{synch}} = \frac{U_r}{B^2}}$$





a
Prompt emission
Afterglow
b
Jet
GRB
Prompt-emission photon
Electron
SSC photon
Synchrotron photon
Synchrotron
SSC
Flux
Energy
©nature