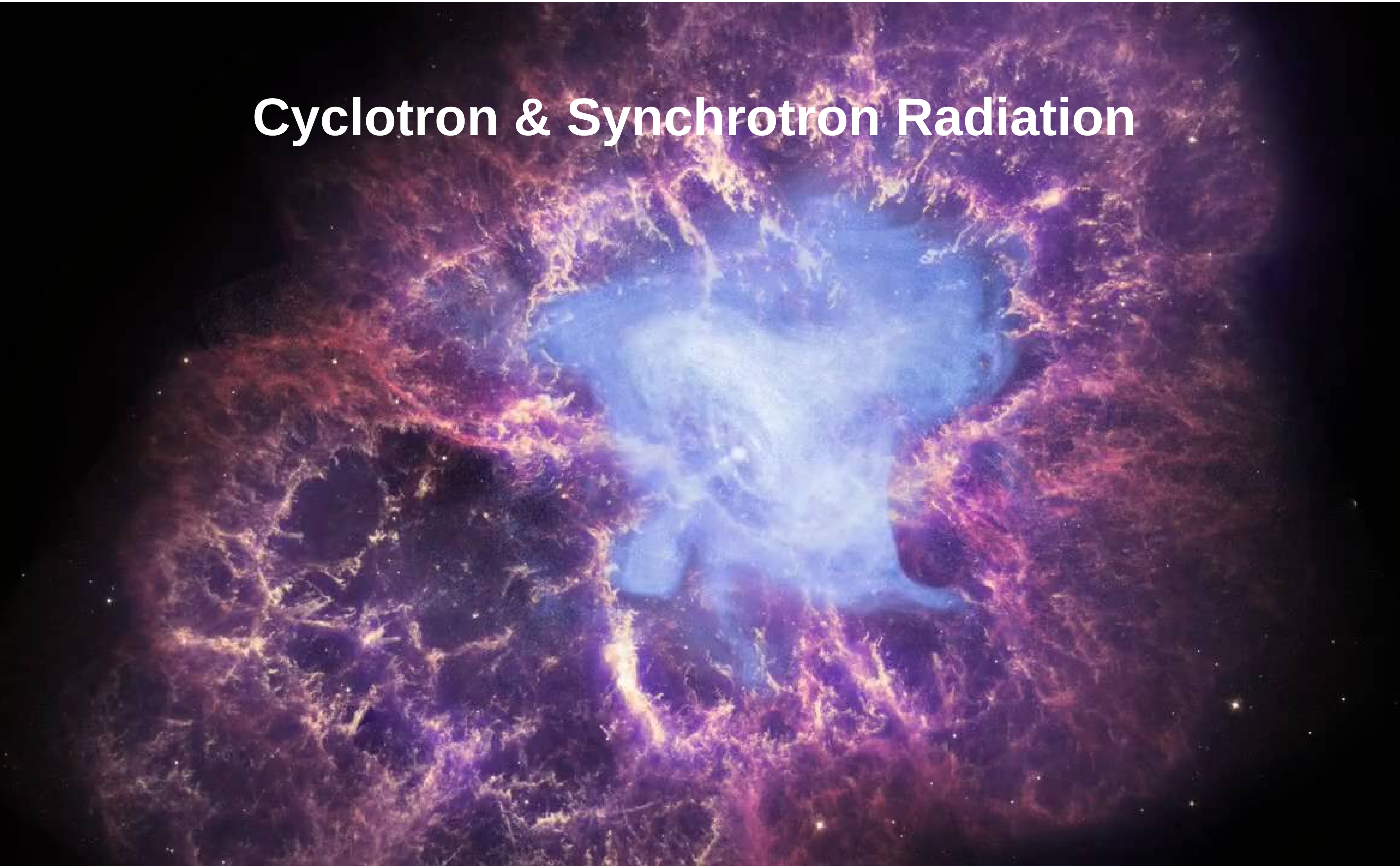
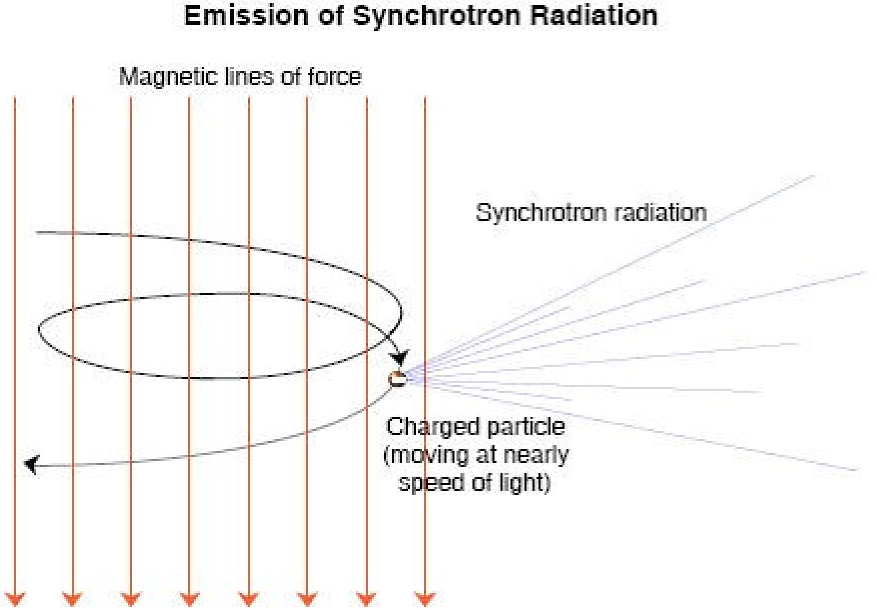
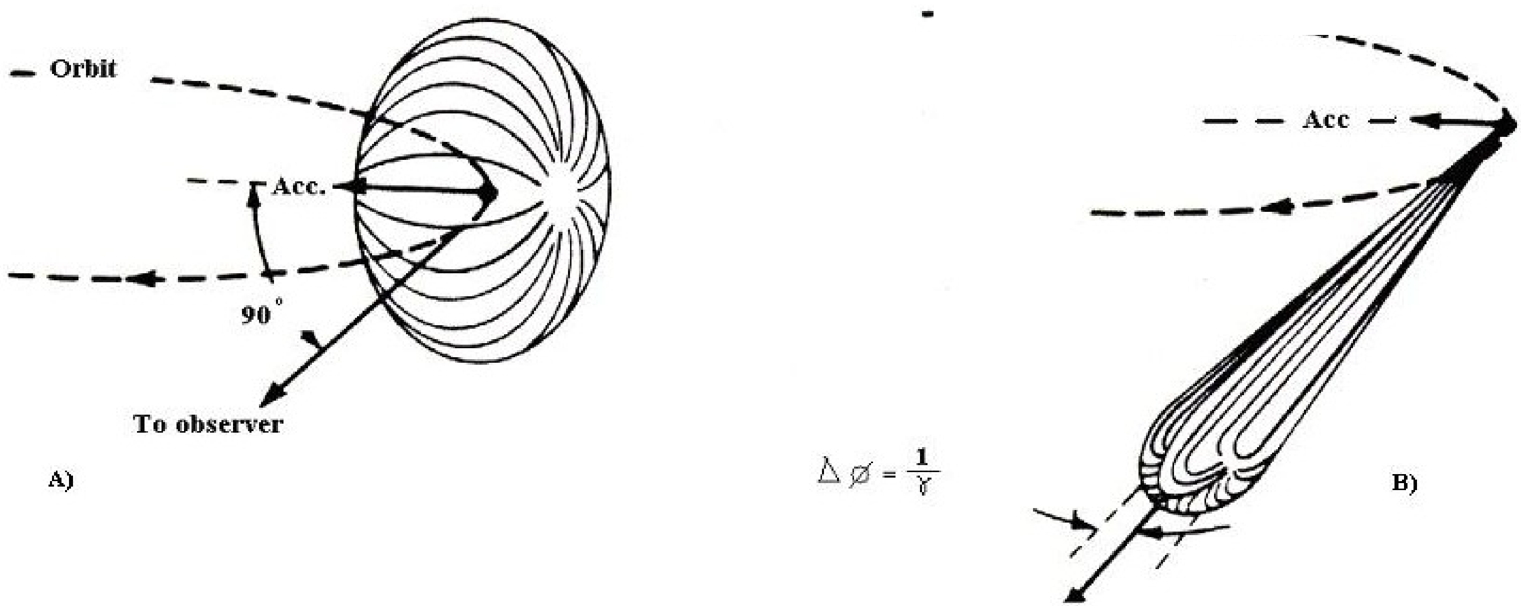
Cyclotron & Synchrotron Radiation



Synchrotron Radiation is radiation emerging from a charge *moving relativistically* that is accelerated *by a magnetic field*.



The relativistic motion induces a change in the radiation pattern which is very collimated (beaming).

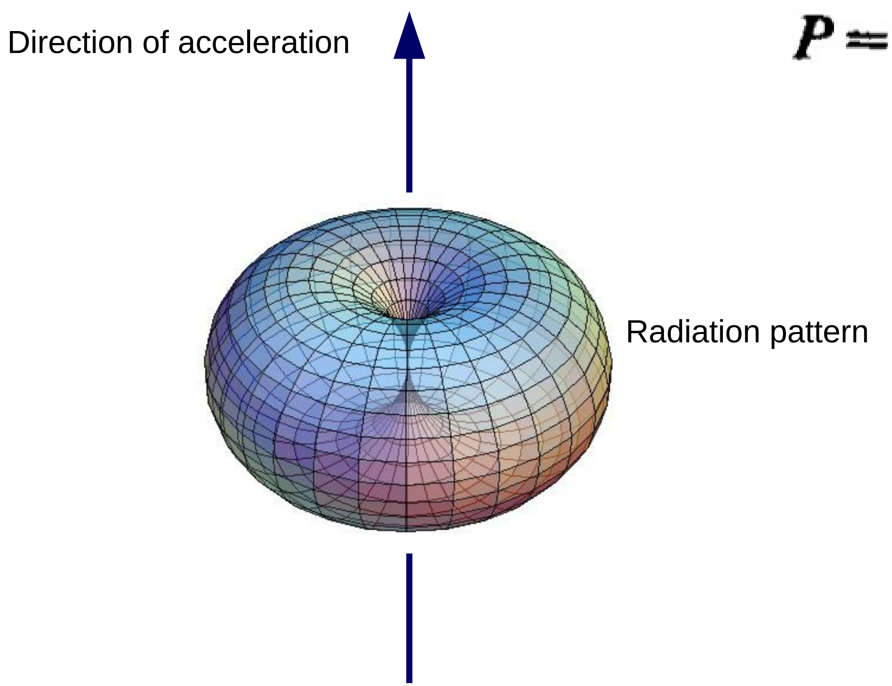


Cyclotron Radiation: power & radiation pattern

To understand synchrotron radiation let’s first begin with the non-relativistic motion of a charge accelerated by a magnetic field.

That the acceleration is given by an electric field, gravity or a magnetic field does not matter for the charge, which will radiate according to the Larmor’s formula

$$P = \frac{2q^2\dot{u}^2}{3c^3}$$



Remember that the radiation pattern is a torus with a sin^2 dependence on the angle of emission:

$$\frac{dW}{dt\,d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta$$

Cyclotron Radiation: cyclotron frequency

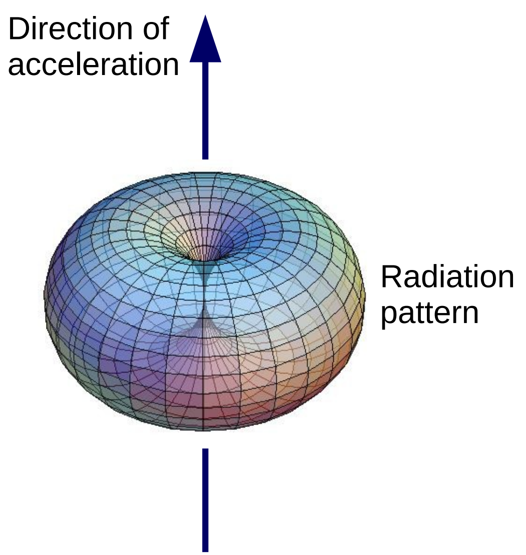
From the angular frequency we can find the period of rotation of the charge:

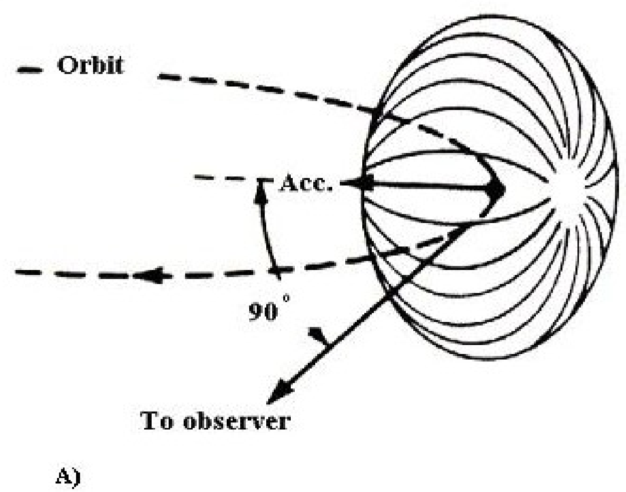
$$T = \frac{2\pi}{\omega_L} = \frac{2\pi m c}{qB}$$

Note that the period of the particle *does not depend on the size of the orbit and is constant if B is constant.*

The charge that is rotating will emit radiation at a single specific frequency:

$$\nu_L = \frac{\omega_L}{2\pi} = \frac{qB}{2\pi m c}$$



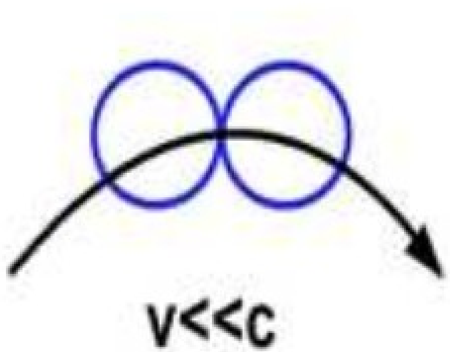


Cyclotron Radiation: power spectrum

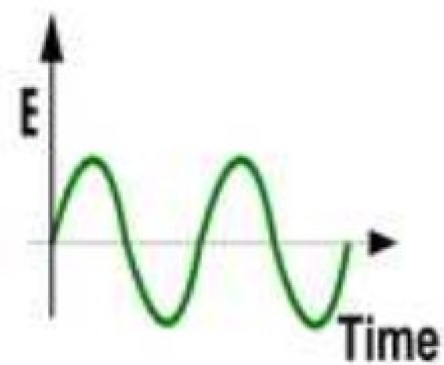
Since the emission appears at a single frequency $\nu_L = \frac{\omega_L}{2\pi} = \frac{qB}{2\pi m c}$

and the dipolar emission pattern is moving along the circle with constant velocity, the electric field measured will vary as a sinusoid and the power spectrum will show a single frequency (the Larmor or cyclotron frequency).

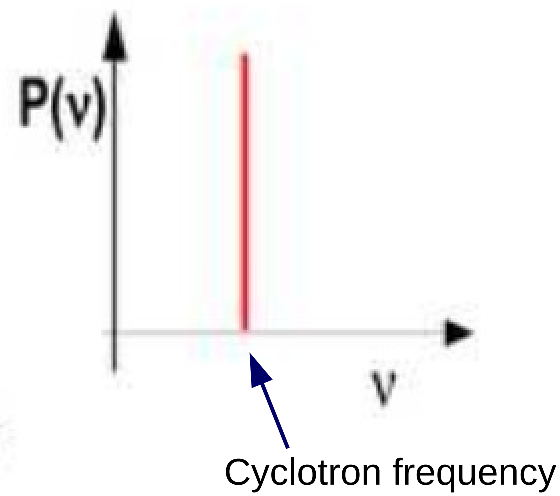
Radiation Pattern



Electric Field E(t)



Power Spectrum P(nu)



Synchrotron Radiation: Power

In the case of cyclotron, as well as (non-rel) Bremsstrahlung we saw that we can use the Larmor's formula to calculate the power emitted by an accelerated charge:

First, let's take the usual rest frame K' with velocity **v** and the lab frame K.

Power is energy over time. Let's Lorentz transform energy and time.

Energy: $dW = \gamma \, dW'$

Note: Here dW and dW' are the energies (in K and K') integrated over the solid angle, not the energy E of a photon. That's why it transforms as written.

Time: $dt = \gamma \, dt'$

Therefore: $P = P'$

So the total power emitted will be a **Lorentz invariant** (but this does not mean that the angular dependence of radiation will be the same of course).

Synchrotron Radiation: Lorentz Transformation of Accelerations

The Larmor's formula in a non-relativistic reference frame is: $P = \frac{2q^2 a^2}{3c^3}$

It is now convenient to split the acceleration in its parallel and perpendicular components to the direction of the rest frame **v**.

Start with the Lorentz transformations of coordinates and velocities:

$$dt = \gamma\left(dt' + \frac{v}{c^2}\,dx'\right) = \gamma\sigma\,dt',$$

$$du_x = \gamma^{-2}\sigma^{-2}\,du'_x,$$

$$du_y = \gamma^{-1}\sigma^{-2}\left(\sigma\,du'_y - \frac{vu'_y}{c^2}\,du'_x\right).$$

where here we have defined for convenience: $\sigma \equiv 1 + \frac{vu'_x}{c^2}$

Therefore:

$$a_x = \frac{du_x}{dt} = \gamma^{-3}\sigma^{-3}\frac{du'_x}{dt'} = \gamma^{-3}\sigma^{-3}a'_x,$$

$$a_y = \frac{du_y}{dt} = \gamma^{-2}\sigma^{-3}\left(\sigma\frac{du'_y}{dt'} - \frac{vu'_y}{c^2}\frac{du'_x}{dt'}\right),$$

$$= \gamma^{-2}\sigma^{-3}\left(\sigma a'_y - \frac{vu'_y}{c^2}a'_x\right)$$

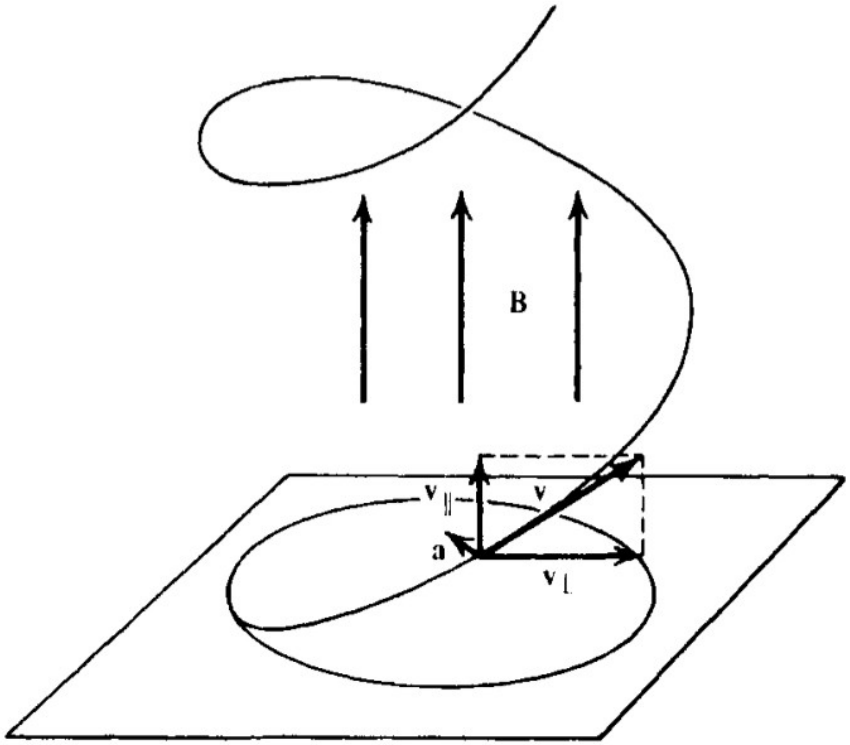
And a similar result holds for the z-component.

Now, if the particle is at rest instantaneously in K', then the velocity $u'_x = u'_y = u'_z = 0.$
Also, sigma = 1 in this case.

Therefore, take the velocity **v** along the x-axis and from the equations above:

$$a'_{\|} = \gamma^3 a_{\|},$$

$$a'_{\perp} = \gamma^2 a_{\perp}.$$



So take now a helical path of a particle in a uniform **B** field. Let's decompose the velocity and accelerations into their parallel and perpendicular components and write the total emitted power:

$$
\begin{aligned}
P = P' &= \frac{2q^2}{3c^3}\left[(a'_\parallel)^2 + (a'_\perp)^2\right] \\
&= \frac{2q^2\gamma^4}{3c^3}(\gamma^2 a_\parallel^2 + a_\perp^2)\ .
\end{aligned}
$$

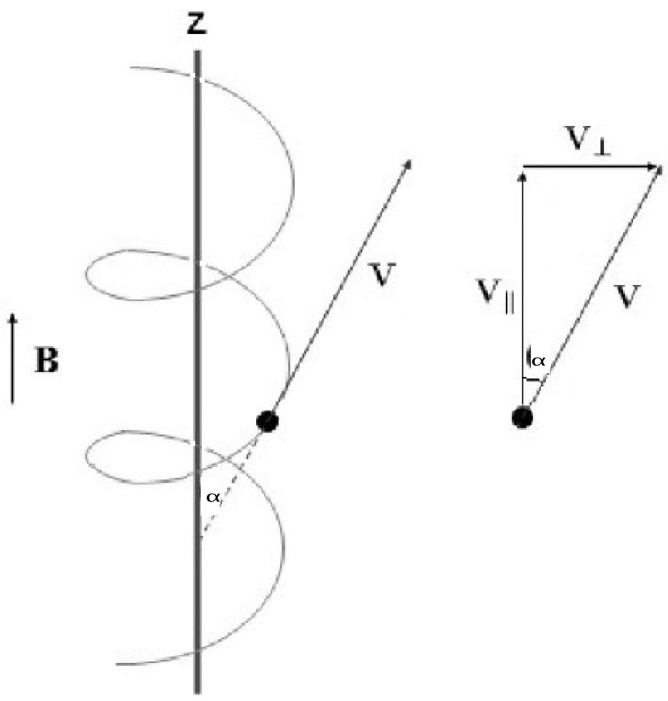
Now from this formula one might suppose that the parallel component is more important than the perpendicular one. This is true when the two accelerations are comparable.
But think for a moment about what happens in a ultra-relativistic beam of particles.
If v~c, you cannot really accelerate the particle in the direction of **v** (the parallel component) whereas with a modest B field you can strongly bend the direction of motion and have a large perpendicular acceleration. So for ultra-relativistic particles the parallel component gives almost zero contribution!

Synchrotron Radiation: Power and Pitch Angle

Since we know the angular frequency of radiation (gyrofrequency): $a_\perp = \omega_B v_\perp$

Substituting this back in the emitted power equation we get:

$$\boxed{P = \frac{2q^2}{3c^3}\gamma^4 \frac{q^2B^2}{\gamma^2 m^2 c^2} v_\perp^2}$$



However, if we have many particles,
each of them will have a different **pitch angle alpha**.
So the perpendicular velocity needs to be averaged over all pitch angles. If we do that, we can write the formula:

$$P = \left(\tfrac{2}{3}\right)^2 r_0^2 c\beta^2\gamma^2 B^2,$$

and rewrite it as:

$$\boxed{P = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B.}$$

Where $\sigma_T = 8\pi r_0^2/3$ is the Thomson cross section
and $U_B = B^2/8\pi$ is the magnetic energy density.

IMPORTANT: The formula written before is valid *only* **for electrons** emitting synchrotron radiation. The reason why we write this formula only for electrons is because in most astrophysical cases you have electron synchrotron. This is because electrons become relativistic much more quickly than protons as they are easier to accelerate.
However using the preceding expression (with the red contour in the previous slide) is fine for any particle of mass m and charge q.

Example: Suppose the protons of the LHC are accelerated up to an energy of 7 TeV and then they are left to cool down due to synchrotron emission. On which timescale do they cool down?

Proton energy: 7 TeV → velocity v~ 29979245800*0.99999999 cm/s
B = 80,000 G
q = 4.8e-10 statC
gamma ~ 7,000
m = 1.67e-24 g

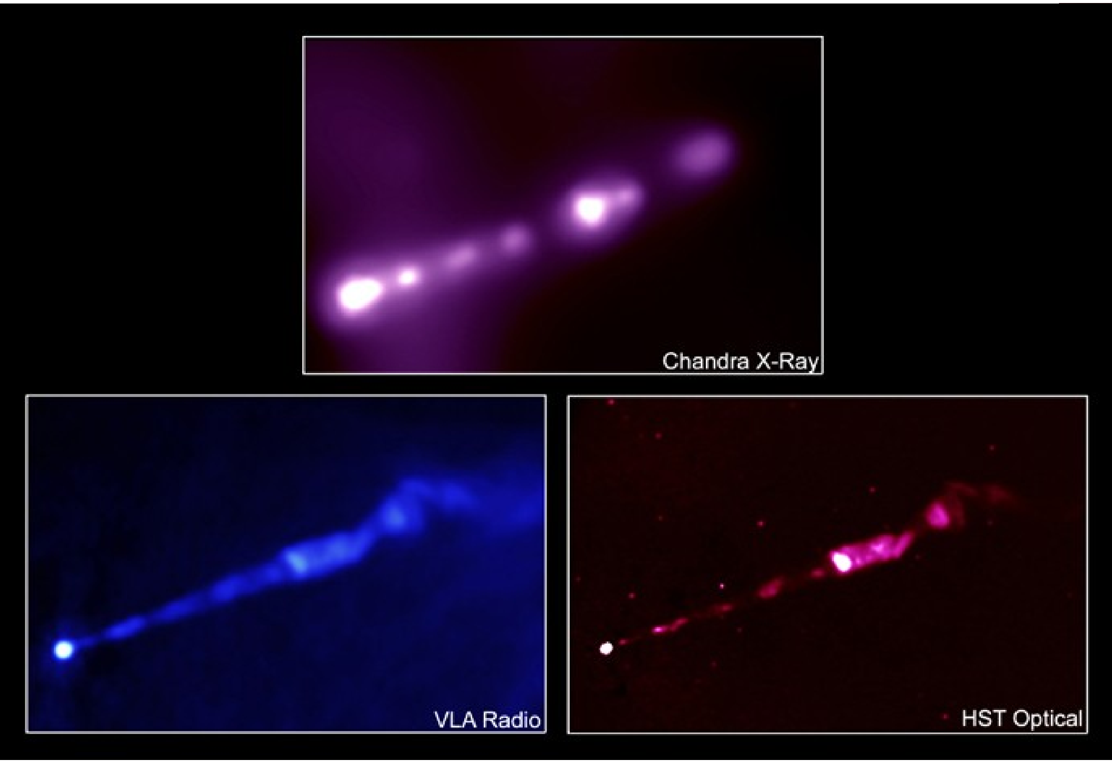
Timescale = (Proton energy) / (synchrotron power) = 2.5 days

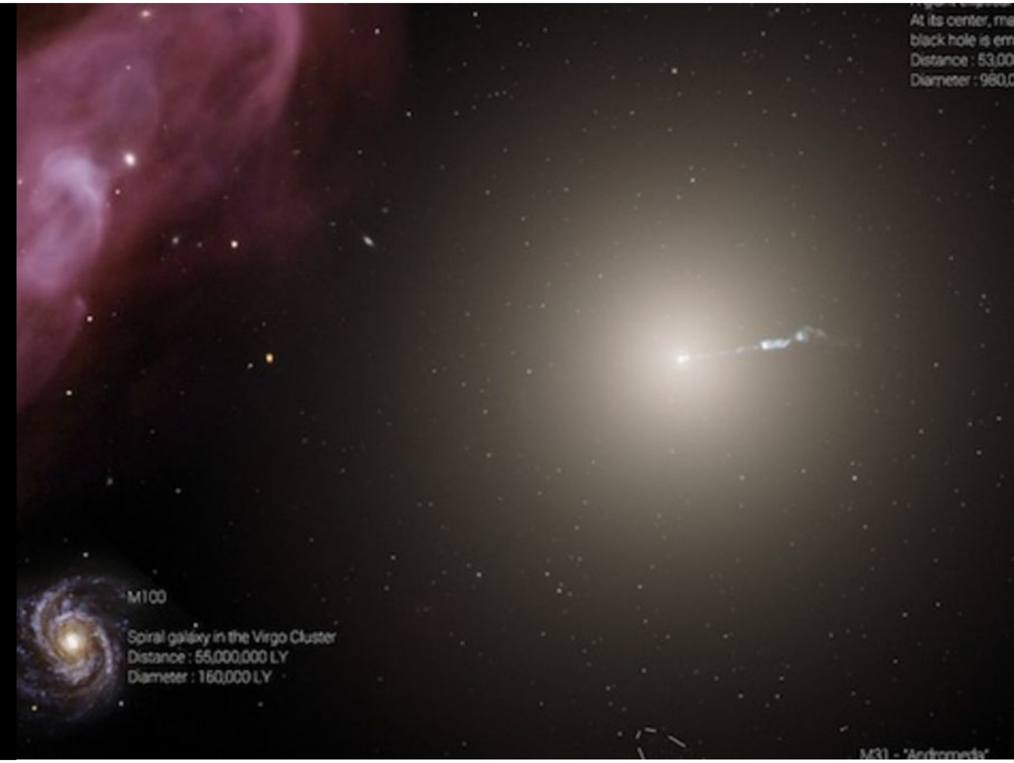
Now let's do the same calculations for a 7 TeV electron (gamma ~ 13,000,000):

Timescale = (Electron energy) / (synchrotron power) = 9 ns

Electrons would cool down a factor 10^13 faster than protons!!

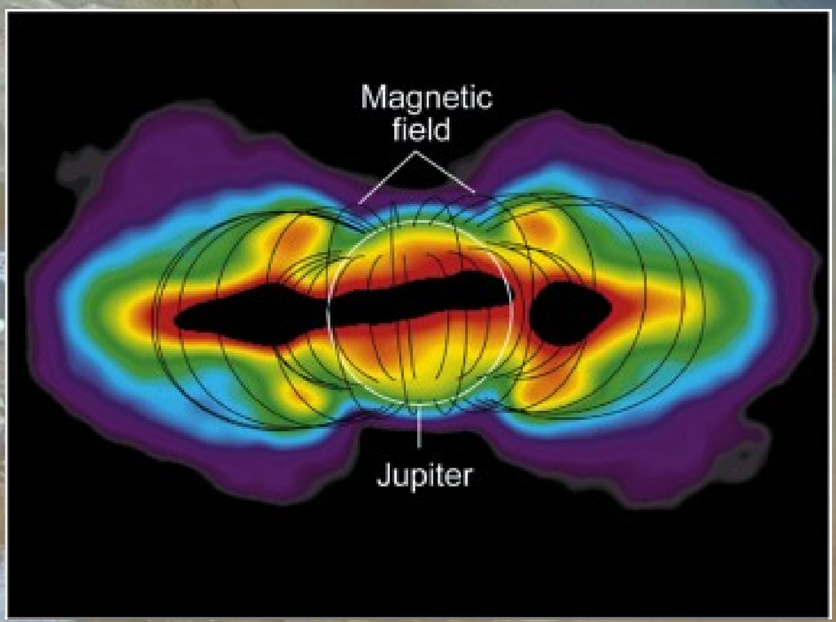
Synchrotron in Astrophysics

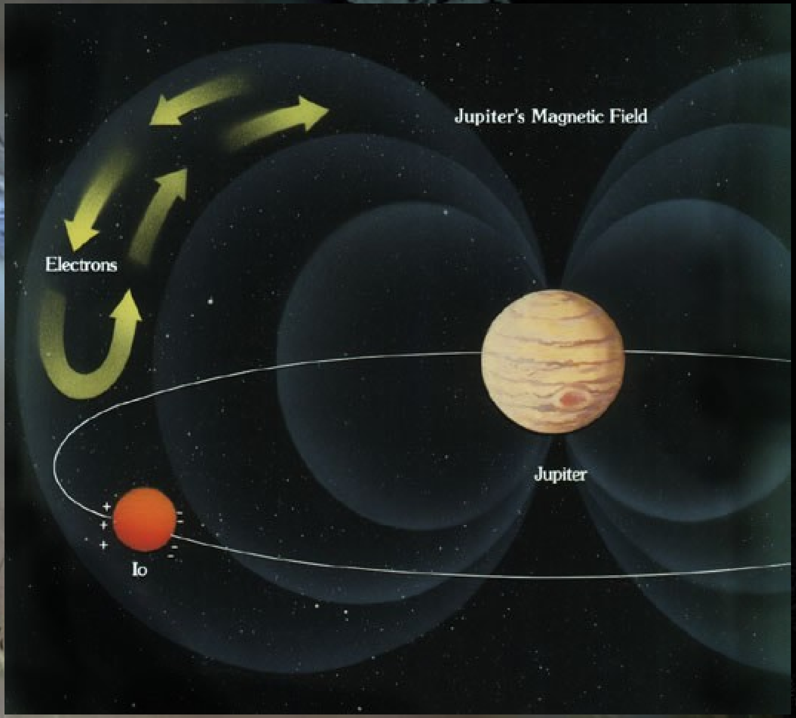




Astrophysical jets are most likely generated by relativistic particles being launched close to a black hole (or even a neutron star when in a binary). Such particles are thought to be electron/positron pairs which then spiral along **B** field lines and generate synchrotron radiation. However, we also know that cosmic rays most likely come from Active Galactic Nucei, where strong **B** fields around supermassive black holes launch streams of ultra-relativistic particles which include protons. So it’s still unclear whether jet emission is due to leptons or hadrons.

Synchrotron Radiation: Jupiter's Belt





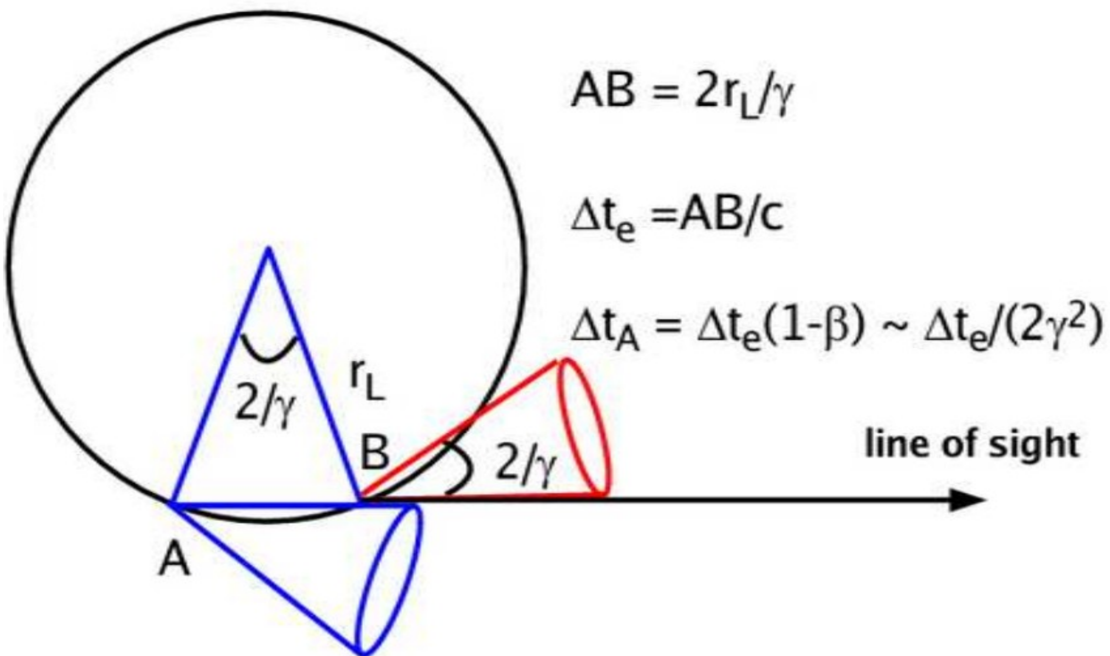
Synchrotron Radiation: Single Particle Spectrum

It's important now to make a distinction between the *emitted* radiation and the *received* radiation. Indeed the received radiation will be such that the observer can see it only when the narrow beam points towards the observer.

This will be a fraction much smaller than the gyration period! The electro(magnetic) field received by the observer will thus be pulsed and therefore it must be composed by many frequencies (think about the Fourier power spectrum of a narrow pulse…).
How small is this period of time? First of all, suppose that we have one electron and the pitch angle is 90 degrees.

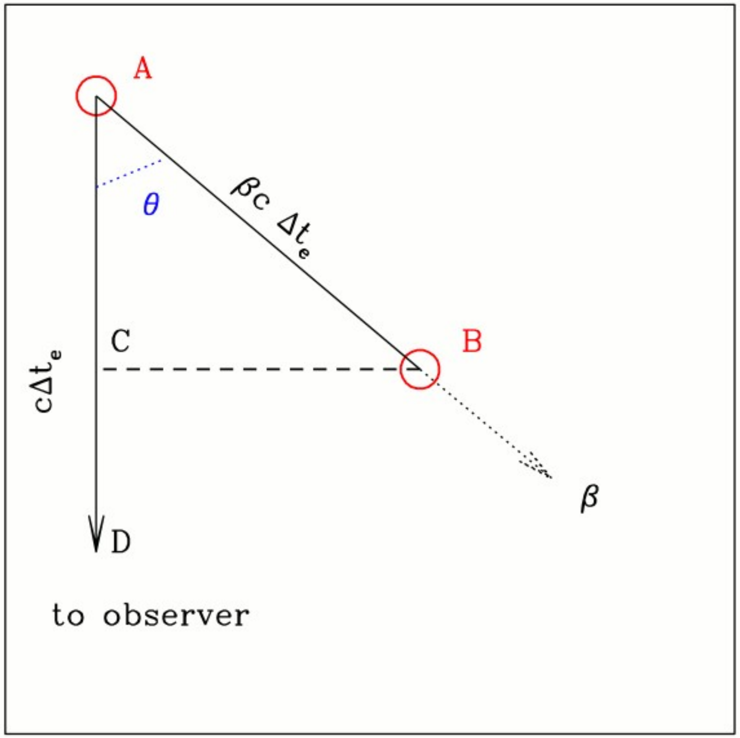
The electron will radiate towards the observer only for a brief amount of time:

$$\Delta t_e \approx \frac{AB}{|\vec{v}|} = \frac{2 r_L}{\gamma |\vec{v}|} = \frac{1}{\pi \gamma \nu_B}$$



Now a question: here we are seeing photons. The expression above is the emission time. What happens when we measure time intervals with photons? Do we have the same result as when we measure time intervals with clocks?

Synchrotron Radiation: Single Particle Spectrum



$$\Delta t_A = \Delta t_e (1 - \beta \cos \theta)$$

Here let's use theta~0 because
we are emitting photons in the same direction of motion:

$$\Delta t_A \approx \Delta t_e (1 - \beta)$$

We can write:

$$\Delta t_{\mathrm{A}} = \Delta t_{\mathrm{e}} (1 - \beta) = \Delta t_{\mathrm{e}} \frac{(1 - \beta^2)}{1 + \beta} \sim \frac{\Delta t_{\mathrm{e}}}{2\gamma^2} = \frac{1}{2\pi\gamma^3 \nu_{\mathrm{B}}}$$

The inverse of this is an angular frequency. We can thus build a **characteristic synchrotron frequency** in this way:

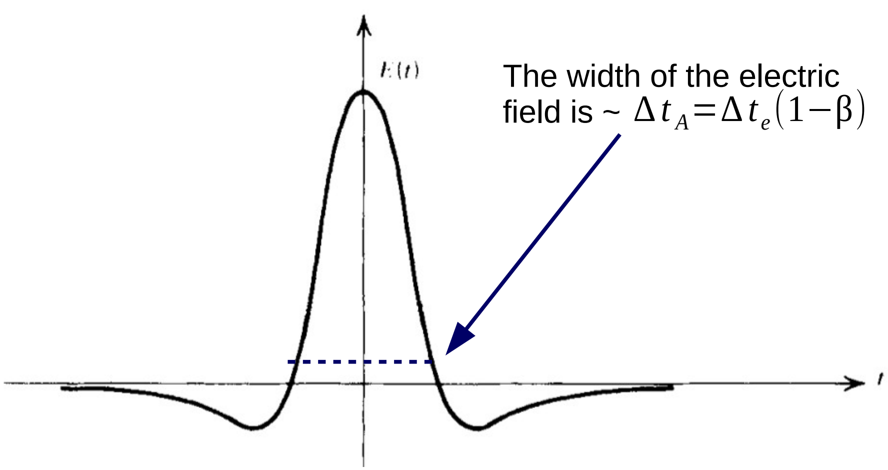
$$\nu_{\mathrm{s}} = \frac{1}{2\pi\Delta t_{\mathrm{A}}} = \gamma^3 \nu_{\mathrm{B}} = \gamma^2 \nu_{\mathrm{L}} = \gamma^2 \frac{eB}{2\pi m_e c}$$

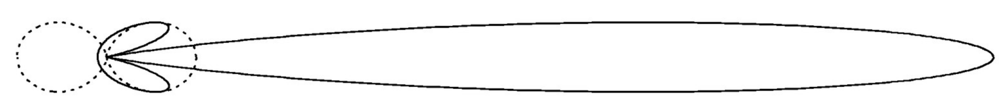
Synchrotron Radiation: Single Particle Spectrum

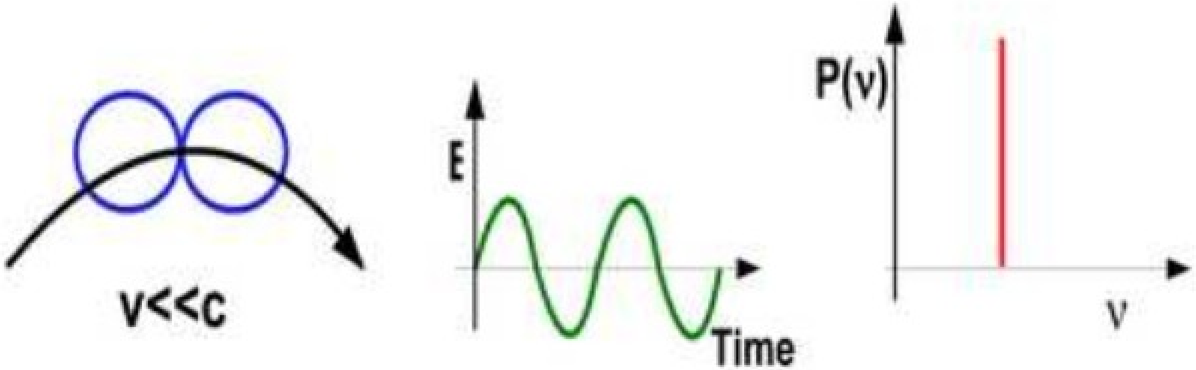
$$\nu_{\mathrm{s}} = \frac{1}{2\pi\Delta t_{\mathrm{A}}} = \gamma^3\nu_{\mathrm{B}} = \gamma^2\nu_{\mathrm{L}} = \gamma^2\frac{eB}{2\pi m_e c}$$

First thing to note: this frequency is gamma^3 larger than the gyration frequency.
This makes sense since we see the emission only for a tiny fraction of the orbit, whereas the gyrofrequency reflects the whole circular motion of the electron.

Second: the *observed* electric field of the rotating charge will be narrow and thus the radiation will spread across many frequencies. The characteristic frequency where the power will drop is of the order of the synchrotron frequency above.

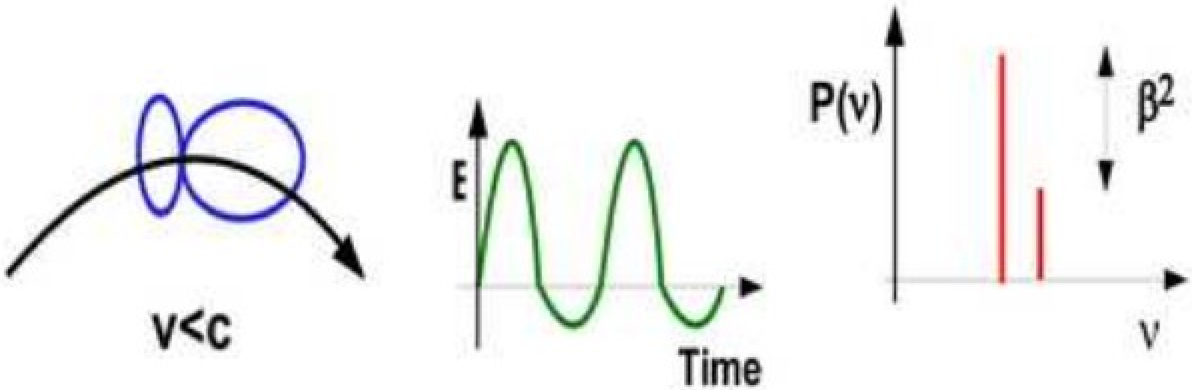






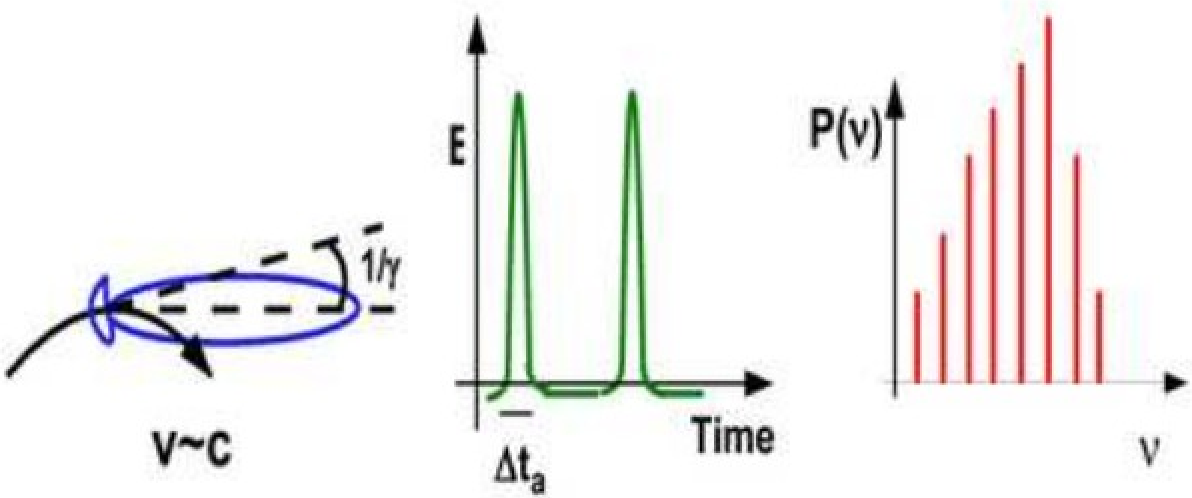
Cyclotron radiation

The charge is moving in a circle, so the electric field variation is sinusoidal



Cyclotron-synchrotron radiation

The charge is moving in a circle, but some aberration of light starts to become visible



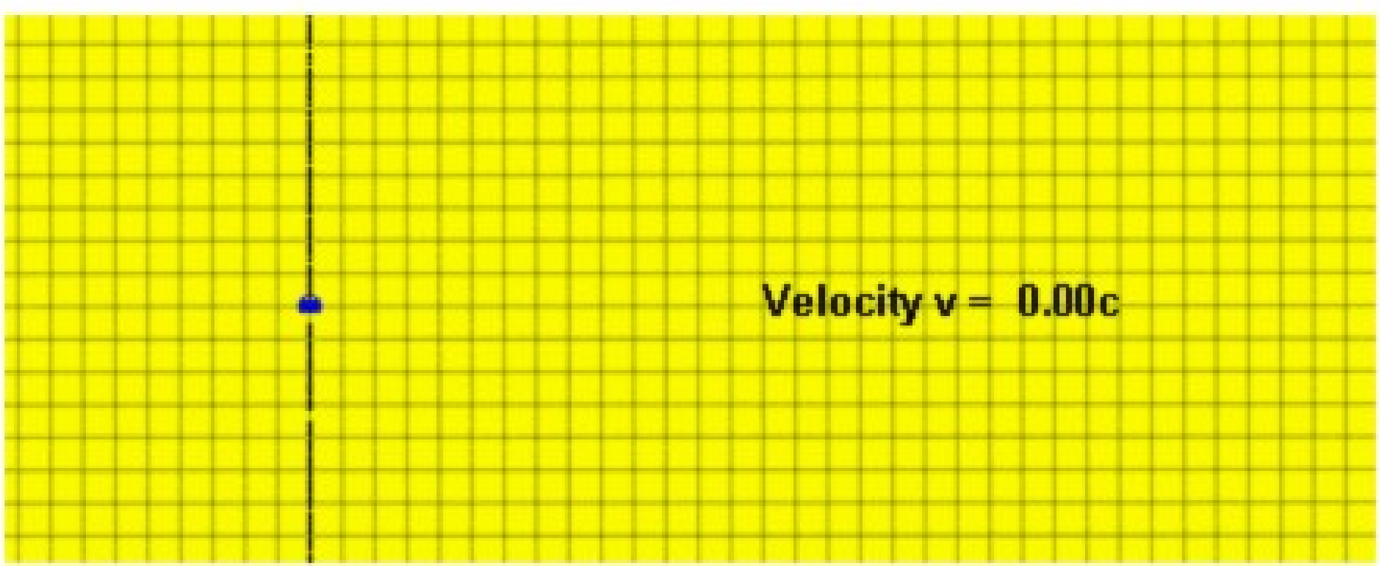
Synchrotron radiation

The charge is moving in a circle, aberration of light is extreme and radiation is seen only for a tiny amount of time when the cone 1/gamma points towards the observer

Synchrotron Radiation: Single Particle Spectrum

In summary: We haven't derived in a rigorous way the single particle spectrum yet, but we can expect the following.

1. The power spectrum will show a broad range of frequencies
2. The width of the power in the power spectrum will be of the order of $\nu_s = \frac{1}{\Delta t_A}$
3. We can expect a quick (exponential) cutoff above these frequencies.
4. The peak of the power must be somewhere around ν_s too, since most of the power is emitted in the interval Δt_A
5. We expect *much more than half* of the power to be emitted within 1/gamma.



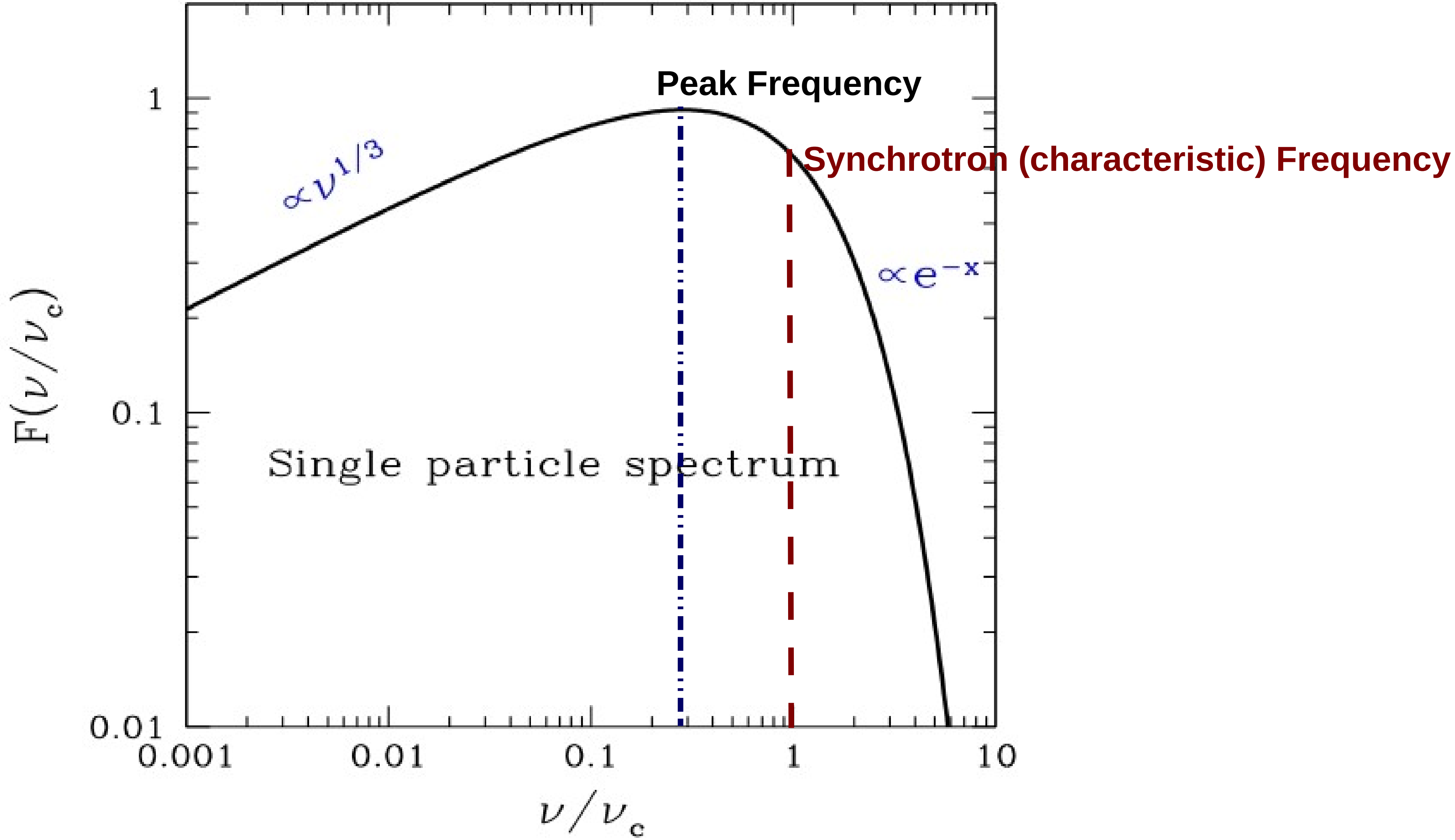
Synchrotron Cooling Time

Once we have the total emitted power we can easily calculate the cooling time of an ensemble of electrons emitting synchrotron as the ratio between the energy E and emitted power P:

$$t_{\rm syn} = \frac{E}{P} = \frac{\gamma m_{\rm e} c^2}{(4/3)\sigma_{\rm T} c U_B \gamma^2 \beta^2} \sim \frac{7.75 \times 10^8}{B^2 \gamma}\ {\rm s} = \frac{24.57}{B^2\gamma}\ {\rm yr}$$

As an example, take a supermassive black hole in an Active Galactic Nucleus.
The magnetic field around the black hole is typically of the order of 1,000 G
The Lorentz factor is also of the order of 1,000, so the electrons cool down on a timescale of just 0.77 seconds.

If instead you calculate the same cooling time very far away from the black hole, where gamma is still 1,000 but the B filed is much smaller (e.g., B~1e-5 G), the cooling time is then of the order of 250 Myr.



Why this synchrotron emission (SINGLE electron) is **not a single frequency?**

We know how the spectrum of one particle emitting synchrotron will look like.
What if we have an ensemble of particles (with energies E between E1 and E2)?

$$N(E)\,dE = C\,E^{-p}\,dE$$

$$N(\gamma)\,d\gamma = C\,\gamma^{-p}\,d\gamma$$

(**C** is a constant that depends on the pitch angle)

We choose a power-law distribution because this is a *very common* case in many different physical, biological and other natural (and man-made) phenomena.

The total power emitted at a specific frequency by our ensemble of (power-law distributed) particles is:

$$P_{tot}(\nu) = C \int_{\gamma_1}^{\gamma_2} P(\nu)\,\gamma^{-p}\, d\nu$$

We substitute the frequency omega with the variable **x**:

$$x = \frac{\nu}{\nu_c}$$

It's important to note that the critical frequency depends on the magnetic field, so the total power emitted will depend on B too.

After some algebra one finds:

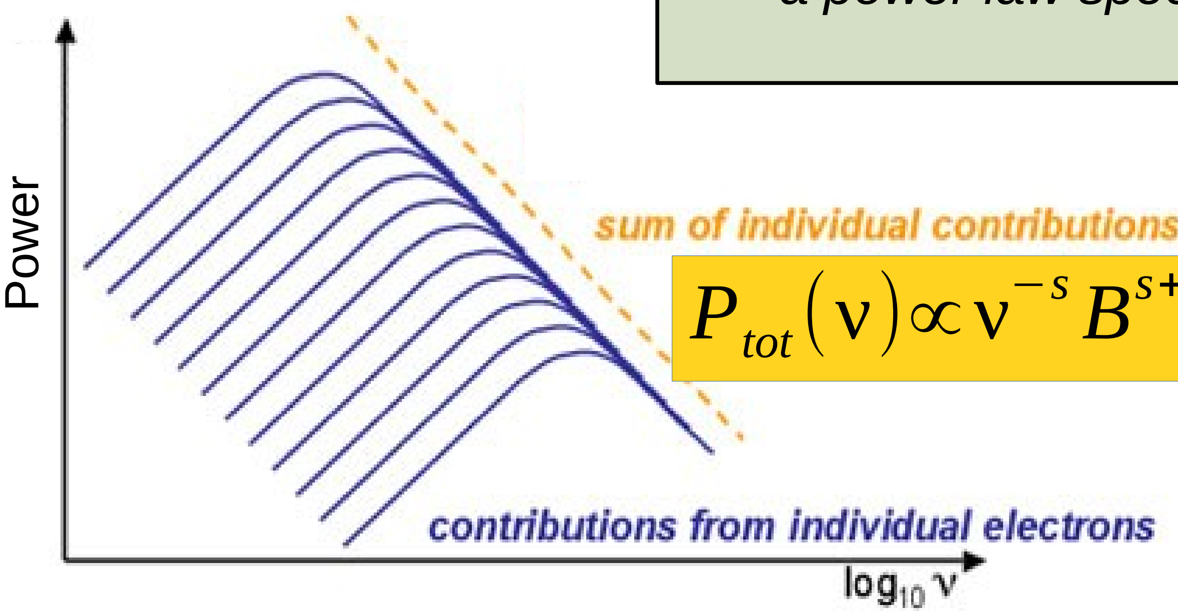
$$P_{tot}(\nu) \propto \nu^{-s} B^{s+1}$$

where we have defined the *spectral index* **s = (p-1)/2**

$$P_{tot}(\nu) = C \int_{\gamma_1}^{\gamma_2} P(\nu)\gamma^{-p} d\nu$$

$$P_{tot}(\nu) \propto \nu^{-s} B^{s+1}$$

Synchrotron:
A power law electron energy distribution produces a power law spectrum

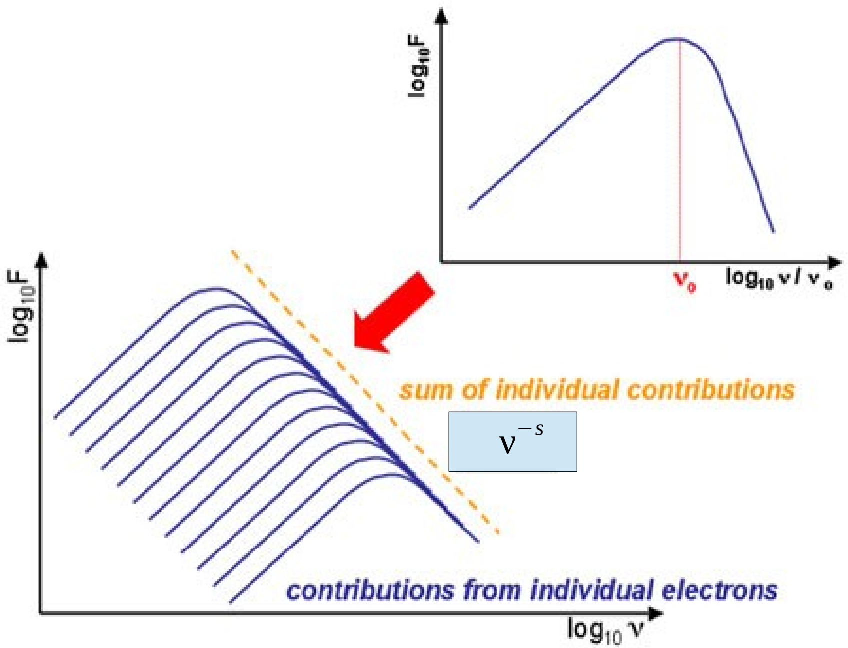


$$P_{tot}(\nu) \propto \nu^{-s} B^{s+1}$$

Self-Absorption

If you noticed, we superimposed different single particle spectra and we said that the initial power-law distribution of electrons (spectral index “p”) gives a power law distribution of emitted radiation (spectral index “s”).

However, according to the principle of detailed balance, to every emission process there is a corresponding absorption process – in the case of synchrotron radiation, this is known as synchrotron self-absorption.



Therefore this power-law with spectral index **“s”** cannot be the whole story...

When does (self)-absorption occur?

Synchrotron Self-Absorption

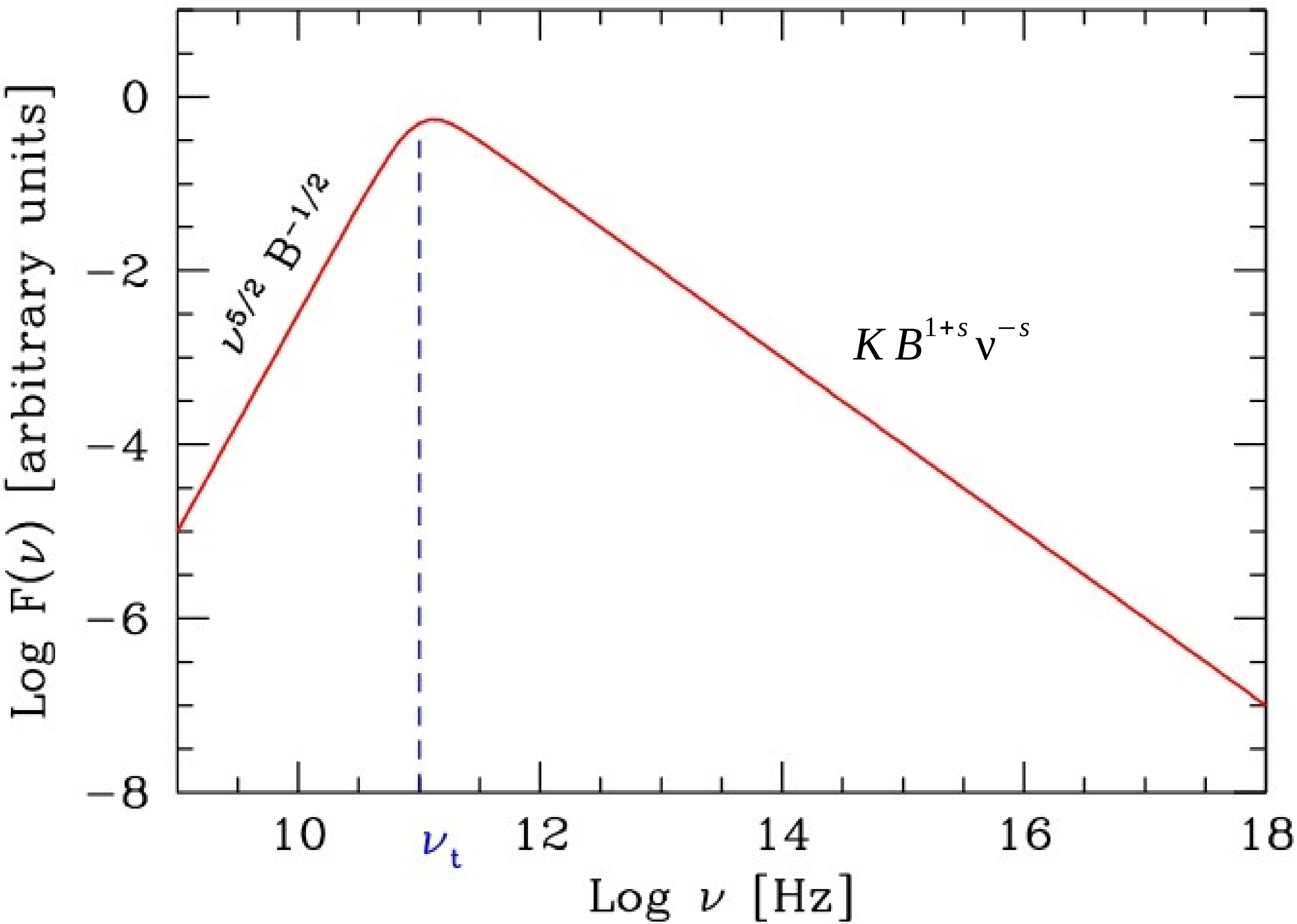
After some algebra, that we will skip here, one finds:

$$\alpha_\nu = \frac{\sqrt{3}\, q^3}{8\pi m}\left(\frac{3q}{2\pi m^3 c^5}\right)^{p/2} C(B\sin\alpha)^{(p+2)/2}\,\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right)\nu^{-(p+4)/2}.$$

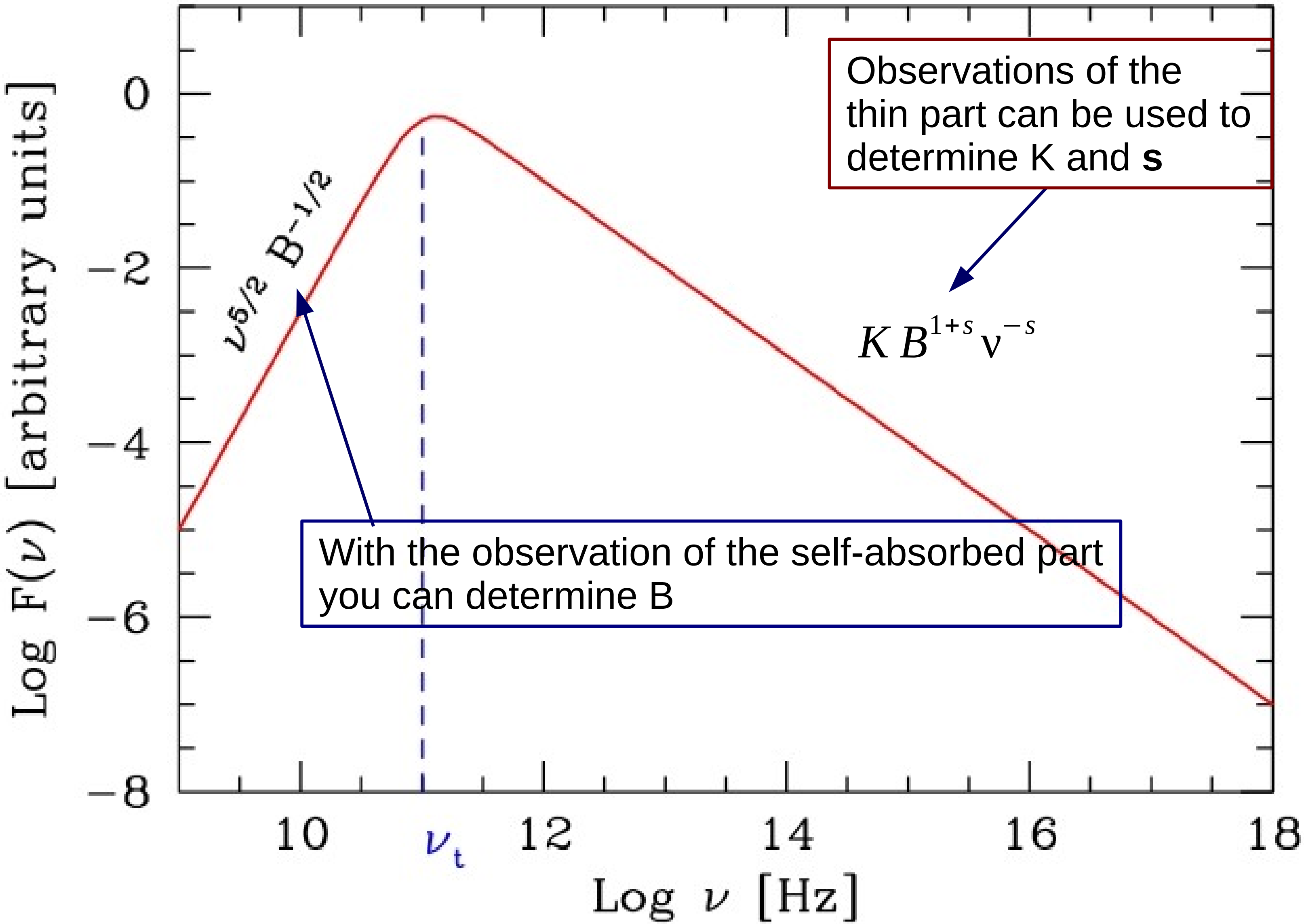
We can write the source function as usual:

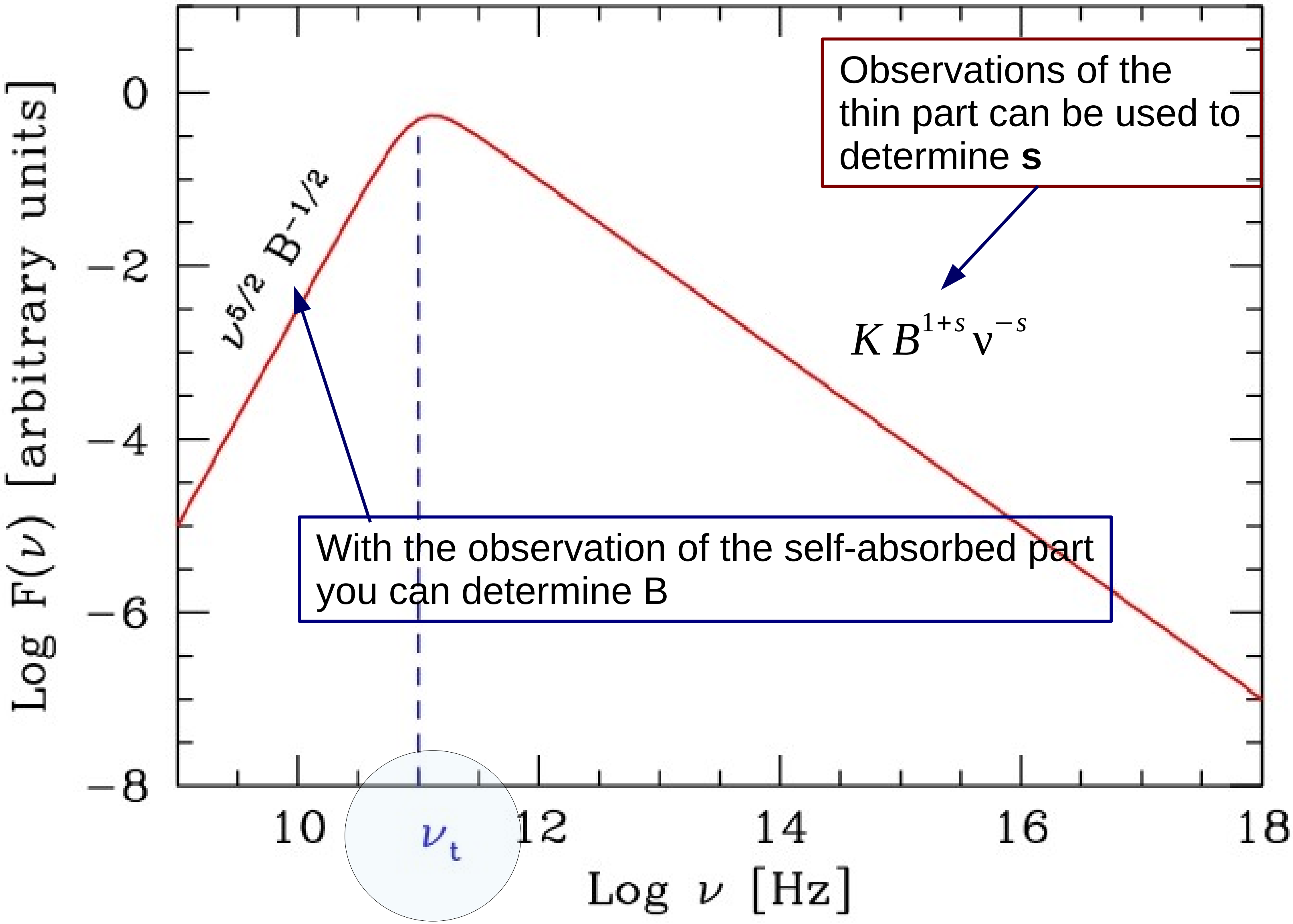
$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{5/2}$$

Note that the slope is NOT 2 as in the Rayleigh-Jeans regime, but it’s 5/2.

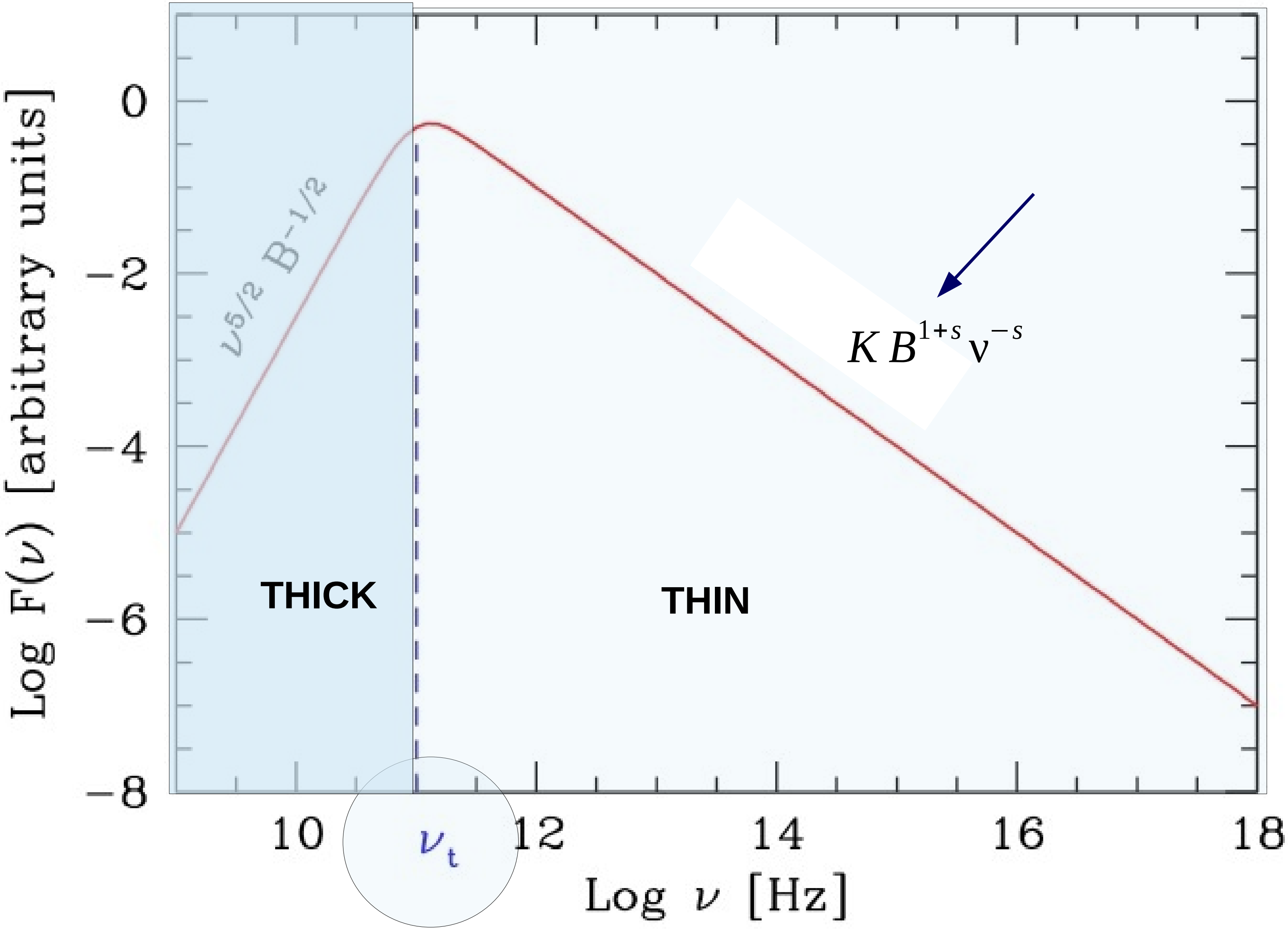


Log F(ν) [arbitrary units]
0
−2
−4
−6
−8
10
12
14
16
18
Log ν [Hz]
ν_t
$\nu^{5/2} B^{-1/2}$
$K B^{1+s} \nu^{-s}$

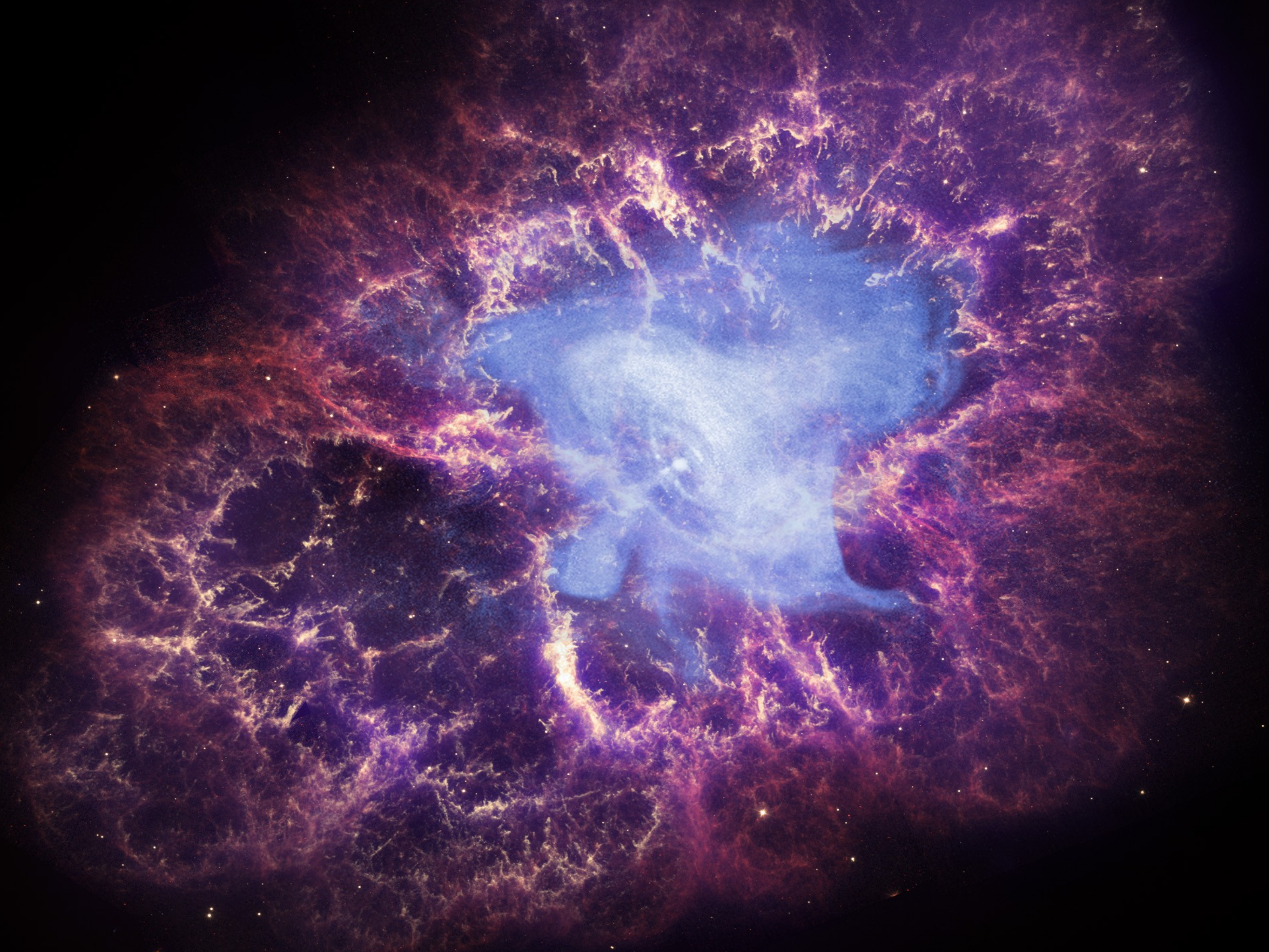

Observations of the thin part can be used to determine K and **s**
$K B^{1+s} \nu^{-s}$
$\nu^{5/2} B^{-1/2}$
With the observation of the self-absorbed part you can determine B
Log F(ν) [arbitrary units]
0
−2
−4
−6
−8
10
ν_t
12
14
16
18
Log ν [Hz]



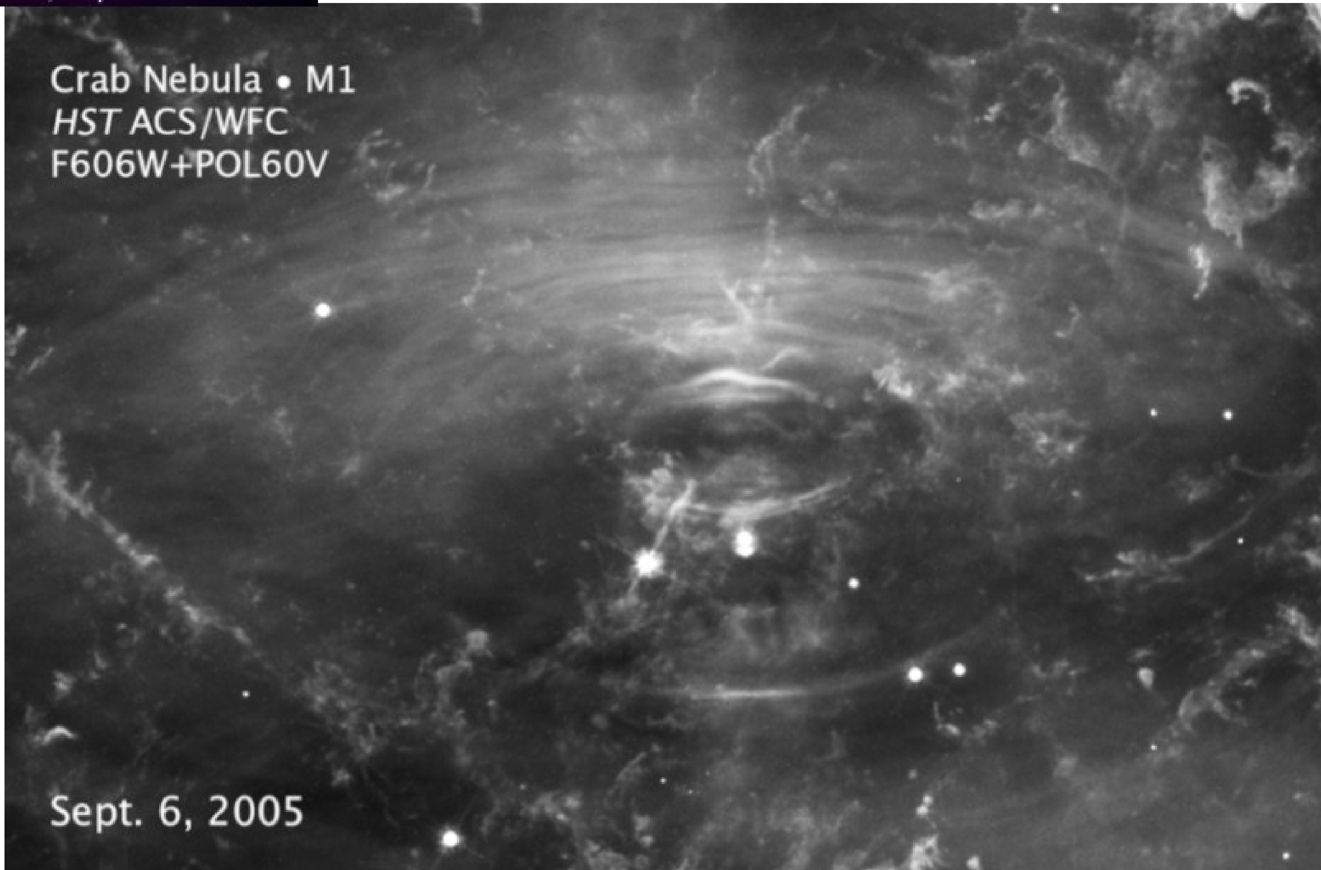
Self absorption frequency: marks the transition from optically thin to thick

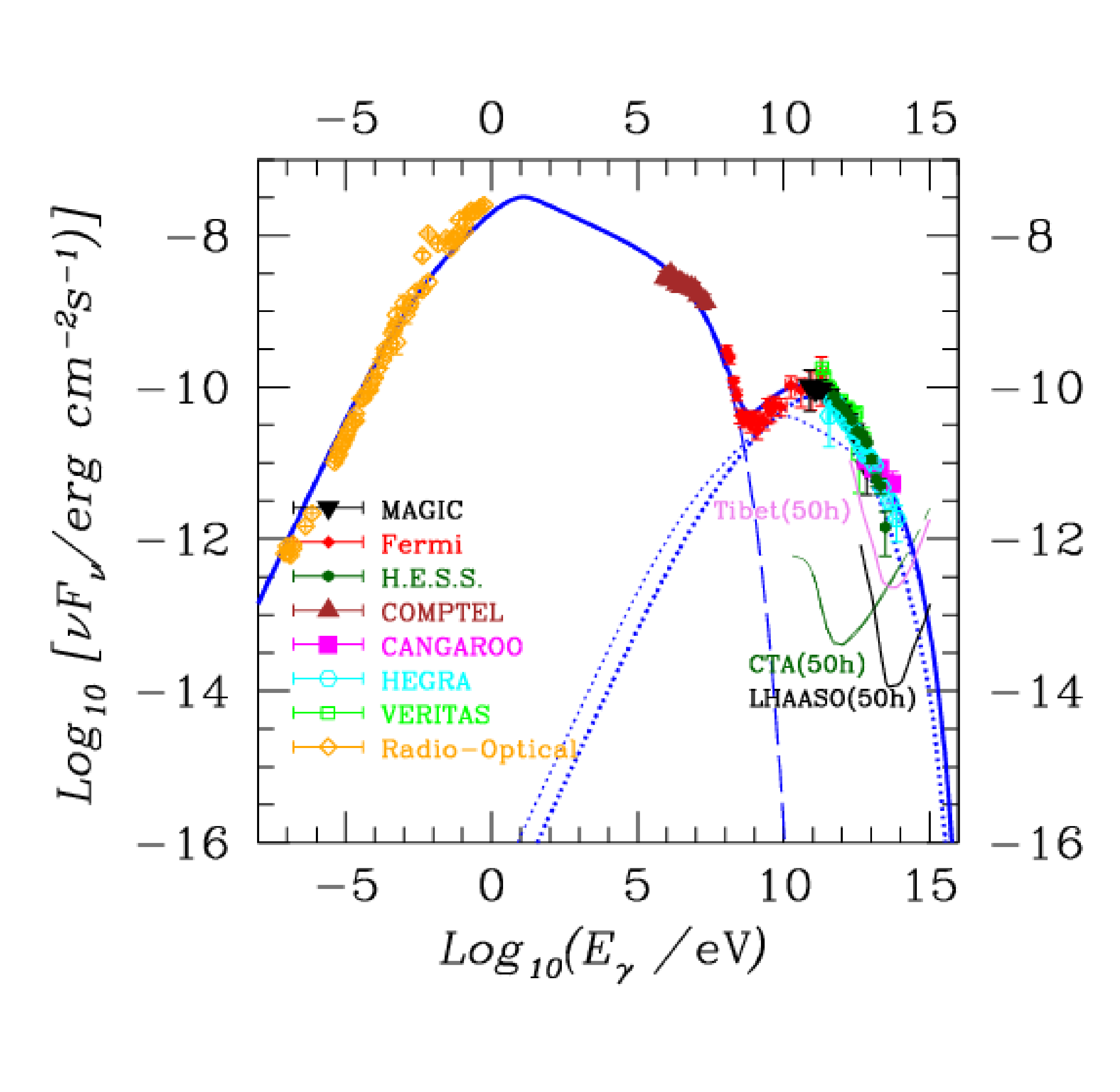


Self absorption frequency: marks the transition from optically thin to thick



Chandra X-Ray Observatory* *(blue) + HST (purple) + Spitzer (red)





−5 0 5 10 15
−8
−10
−12
−14
−16
$Log_{10}\ [\nu F_\nu / erg\ cm^{-2} s^{-1})]$
$Log_{10}(E_\gamma\ /eV)$
MAGIC
Fermi
H.E.S.S.
COMPTEL
CANGAROO
HEGRA
VERITAS
Radio−Optical
Tibet(50h)
CTA(50h)
LHAASO(50h)