Norwegian University of Science and Technology

PION CONDENSATION IN QCD

From χPT to weak coupling

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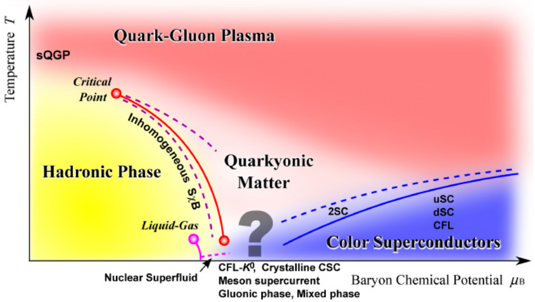
References: Phys. Rev. D 109, 034022 (2024), Phys. Rev. D 111, 034017 (2025), Phys. Rev. D 113 014026 (2024).

Introduction

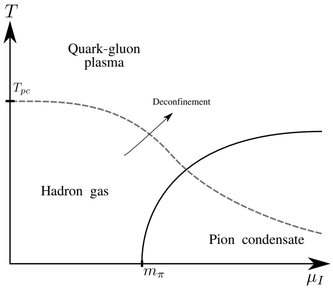
- QCD phase diagram [2]

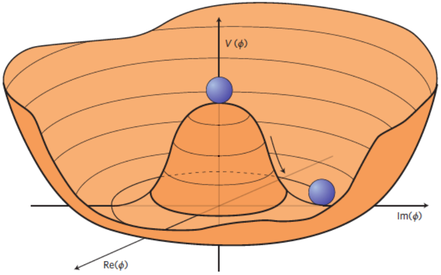


- Strong magnetic field B - no sign problem. Amenable to lattice simulations
- Finite isospin chemical potential μ_I - no sign problem. Amenable to lattice simulations (talk by Sagun)

[2] Fukushima and Hatsuda (2011)

Pion condensation at finite isospin chemical potential





- This talk: BEC at finite μ_I and $T = 0$
- Pion condensation at $\mu_I^c = m_\pi$ ($U(1)_{I_3}$ broken. π^+ condenses)
- Behavior for $\mu_I \simeq m_\pi$?
- Behavior at large μ_I?

Chiral perturbation theory

- Chiral perturbation theory is a low-energy theory for QCD based on its global symmetries and degrees of freedom [3]
- $SU(2)_L \times SU(2)_R$ for two flavors (pions). $SU(3)_L \times SU(3)_R$ for three flavors (pions, kaons, and eta)

$$
\begin{aligned}
\mathcal{L}_2 &= \frac{1}{4}f^2\langle\nabla^\mu\Sigma^\dagger\nabla_\mu\Sigma\rangle + \frac{1}{4}f^2m^2\langle\Sigma+\Sigma^\dagger\rangle \ , \\
\Sigma &= e^{i\tau_a\phi_a/f} \ , \qquad \nabla_\mu\Sigma \equiv \partial_\mu\Sigma - i\left[v_\mu,\Sigma\right] \ , \\
v_\mu &= \delta_{\mu,0}\left(\tfrac{1}{3}\mu_B + \tfrac{1}{2}\mu_I, \tfrac{1}{3}\mu_B - \tfrac{1}{2}\mu_I\right) \ , \\
f &\sim f_\pi \qquad\qquad m \sim m_\pi
\end{aligned}
$$

- Results independent of μ_B

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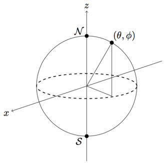
[3] Gasser and Leutwyler '84 and '85

Rotated ground state and pion condensation

- Ground state [4]

$$\Sigma_\alpha = e^{i\alpha\tau_1} = \mathbb{1}\cos\alpha + i\tau_1 \sin\alpha \ .$$



- Static Hamiltonian

$$\begin{aligned}\mathcal{H}_{\rm static} &= -\frac{1}{4}f^2\langle[v_\mu,\Sigma^\dagger][v^\mu,\Sigma]\rangle - \frac{1}{4}f^2m^2\langle\Sigma^\dagger+\Sigma\rangle \\ &= -f^2m^2\cos\alpha - \frac{1}{2}f^2\mu_I^2\sin^2\alpha \ .\end{aligned}$$

- Competition between the two terms

$$\begin{aligned}\cos\alpha &= \frac{m^2}{\mu_I^2}\ , \quad \mu_I^2 \geq m^2 \ , \\ \alpha &= 0\ , \quad \mu_I^2 \leq m^2 \ .\end{aligned}$$

[4] Son and Stephanov 01

Calculations and some key results

- Calculate thermodynamic potential $\Omega(\mu_I, \alpha)$ in a low-energy expansion
- NLO calculation: one-loop from $\mathcal{L}_{\rm LO}$ and static terms from $\mathcal{L}_{\rm NLO}$. Requires renormalization of couplings and their experimental values
- Silver-Blaze property
- Second-order phase transition exactly at the physical pion mass, $\mu_I^c = m_\pi$
- Reproducing pressure, energy density, damping rate...for the dilute Bose gas [5]
- Calculation of quark and pion condensates
- Speed of sound approaches c as $\mu_I \to \infty$. Wrong degrees of freedom

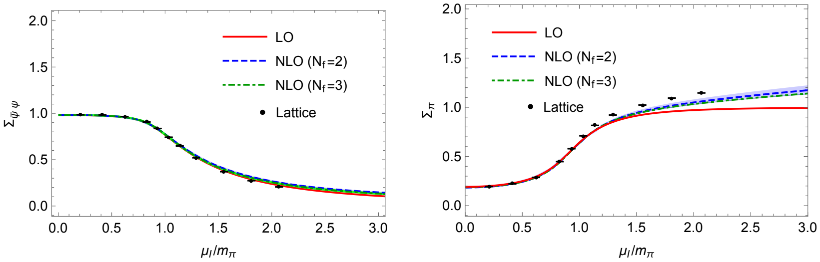
[5] Bogoliubov '47, Lee, Huang, and Yang '57, Wu, Hugenholz and Pines, Sawada '59,. Braaten and Nieto '97.

Quark and pion condensates

- Calculate Ω in presence of pionic source (in addition to quark mass m_q).

$$\langle\bar{\psi}\psi\rangle = \frac{\partial\Omega}{\partial m_q} = \langle\bar{\psi}\psi\rangle_0 \cos\alpha + \ldots$$

$$\langle\pi^+\rangle = \frac{\partial\Omega}{\partial j} = \langle\bar{\psi}\psi\rangle_0 \sin\alpha + \ldots$$



- Lattice results by Brandt et al '18–'22 (here with finite source j)

Quark-meson model

- Including quark degrees of freedom

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{2}(\partial_\mu\pi_0)(\partial^\mu\pi_0) + (\partial_\mu + i\mu_I\delta_{\mu 0})\,\pi^+\left(\partial^\mu - i\mu_I\delta^{\mu 0}\right)\pi^- \\
& -\frac{1}{2}m^2(\sigma^2+\vec{\pi}^2) - \frac{\lambda}{24}(\sigma^2+\vec{\pi}^2)^2 + h\sigma + \bar{\psi}(i\partial\!\!\!/ + \gamma^0\hat{\mu})\psi - g\bar{\psi}[\sigma + i\gamma^5\vec{\pi}\cdot\vec{\tau}]\psi
\end{aligned}
$$

- Include quark loop, treat bosons at tree level
- Determine the parameters using the on-shell scheme in the same approximation as Ω
- Calculate the thermodynamic potential Ω, pressure, and energy density

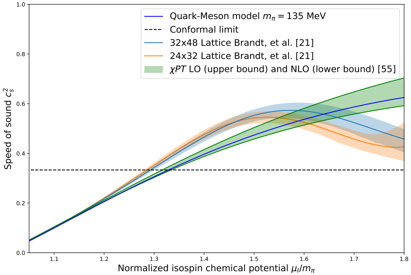
Some analytical results

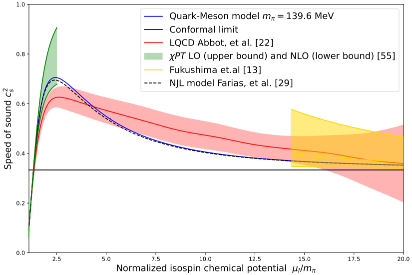
1. Second order phase transition exactly at $\mu_I^c = m_\pi$
2. Large-μ_I behavior

$$
\begin{aligned}
g_0^2\bar{\rho}_0^2 &= m_q^2 \exp\left[\frac{(4\pi)^2}{4N_c g_0^2} - 1 - F(m_\pi^2) - m_\pi^2 F'(m_\pi^2)\right] , \\
p &= \frac{N_c\mu_I^4}{6(4\pi)^2} + \frac{2N_c\mu_I^2 g_0^2\bar{\rho}_0^2}{(4\pi)^2} , \\
\epsilon &= \frac{N_c\mu_I^4}{2(4\pi)^2} + \frac{2N_c\mu_I^2 g_0^2\bar{\rho}_0^2}{(4\pi)^2} , \\
c_s^2 &= \frac{1}{3}\left[1 + \frac{4g_0^2\bar{\rho}_0^2}{\mu_I^2}\right] .
\end{aligned}
$$

Speed of sound

1. Good agreement for small μ_I.[6] $c \to 1$ as $\mu_I \to \infty$ in χpt
2. Quark-meson model [7] shows qualitatively same behavior as lattice data for all μ_I and weak coupling at large. [8]. Excellent agreement with the NJL model. [9]





[6] Brandt et al '18-'22.
[7] Kojo '23, Brandt et al '25, JOA and M. Nødtvedt '25, Ayala et al '25
[8] Abbot et al '24. Fukushima and Minato '25 (Poster)
[9] Farias et al '25 (talk Friday)

Conclusions and outlook

- Conclusions

  1. First calculations of speed of sound, quark and pion condensates at NLO. Good agreement with lattice
  2. Reduces to the dilute Bose gas in nonrelativistic limit with a phonon damping rate $\mathcal{O}(p^5)$ for small p
  3. BCS phase not described within χPT.
  4. The speed of sound relaxes to the conformal limit from above

- Outlook

  1. Including diquarks and color superconductivity (2SC and CFL) with applications to stars (poster by Nødtvedt)
  2. Finite temperature