Intro to Thermal Field Theory (w/ a dense focus)

- NOT TFT = Topological Field Theory
- 60 min crash course focussing on introducing tools
- Assuming some QFT & Stat Mech basics
- ASK QUESTIONS.

No coffee until 16:15 >:(

o o O And think about the bubbles

Applications

x Neutron stars

Microphysics of EoSs, Transport

- Early Universe Cosmo
- StatPhys / Condensed matter ("Statistical field theory")
- Heavy-ion Collisions
- Fun maths/mathphys :)

ASTRO-INCLINED FOLKS BEWARE:

$$\hbar = k_B = c \equiv 1.$$

What is thermal FT?

- Quantum (or statistical) fields in a medium

↳ Intrinsic properties

- Temperature T
- Chemical potentials μ_i
- External fields $B, E \dots$
- ...

all already defined

complications:

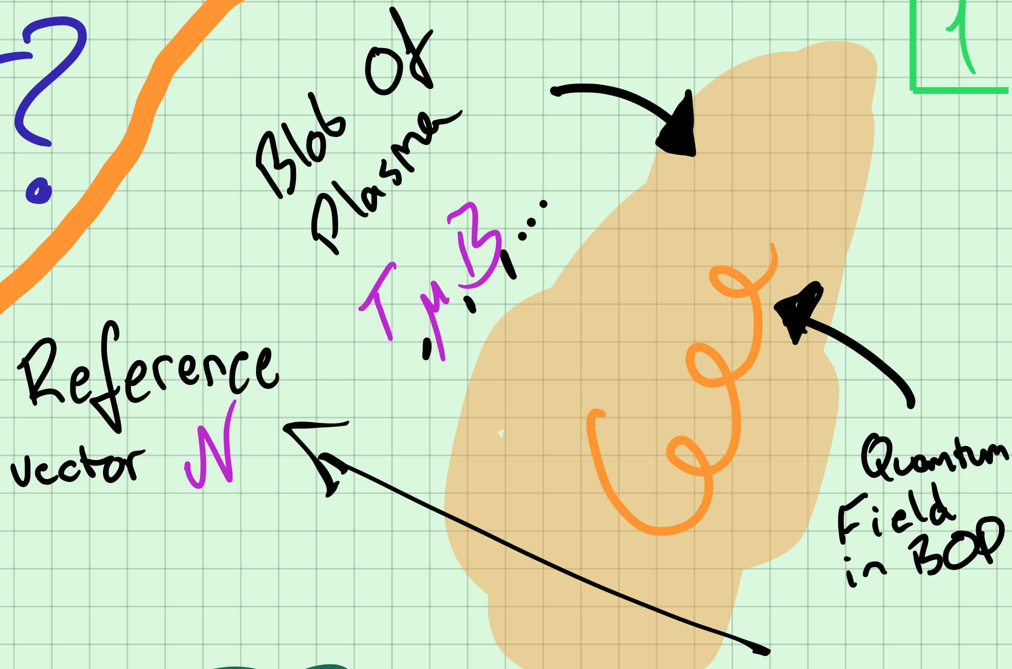
- New intrinsic scale(s)

  [cf: ϕ^4 w/ two masses instead of one]
- Reference frame of the medium

  [~cf: adding extra external momentum]

  ↳ $SO(d,1) \hookrightarrow SO(d) \to$ complicates tensors
- IR problems from soft mom. scales

Extra Complications



How is QFT modified in a medium?

This lecture

How do QFs form a thermal medium?

A highly nontrivial open problem

What is finite density?

...or finite chemical potential

Neutron stars are pretty dense, So we'll focus on this.

Which conservation law for Neutron Stars?

Recall your stat mech (or chemistry): Systems w/ particle species transmutating is in a Gibbs ensemble with a measure

$$\exp(-\beta H + \mu Q)$$

Hamiltonian

Number Density $Q = \int_{space} d^d x \, j^0$

Noetherian symmetry current

Emmy, ca. 1900 - 1910

Why is finite μ ⟺ Presence of a Noether charge?

Because finite μ ≡ excess of stuff (type of particles, quantum number...)

equilibrium → Q constant → $\operatorname{div} j = 0$

⇝ Global symmetry of the Lagrangian

e.g. U(1) sym on fermions

$$\bar{\psi} \mapsto e^{-i\theta} \psi \Rightarrow j_\alpha = \bar{\psi} \gamma_\alpha \psi \Rightarrow Q = \int_{space} \psi^\dagger \psi$$

A trick to thermal fields

Given a QFT, how do we describe it @ finite T, μ?

If we only care about Static Quantities like the pressure, this is easy:

× Change from

Minkowski

$$\mathbb{R}^{d,1} \longrightarrow S^1_{1/T} \times \mathbb{R}^d$$

Euclidean

"thermal compactification"

(time)

× Shift derivatives acting on fields w/ finite chemical potentials

$$F[\partial_0, \ldots] \psi_f \mapsto F[\partial_0 - \mu_f, \ldots] \psi_f$$

Derived from $-\mu Q$ of last slide

Medium modifies its geometry!

Zero-components of momenta:

[Recall Fourier analysis]

$$\underbrace{p_0 \mapsto \begin{cases} 2\pi n T & \leftarrow \text{Bosons} \\ \pi(2n+1)T, \ n \in \mathbb{Z} & \leftarrow \text{fermions} \end{cases}}_{\text{Matsubara Mode}}$$

integrals $\displaystyle\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \int_{\mathbb{R}^d} \frac{d^dp}{(2\pi)^d} \mapsto T \sum_{p_0} \int_{\vec{p}} \equiv \sum\!\!\!\!\!\!\int$

$$\underbrace{p_0 \mapsto p_0 - i\mu_f}$$

Complexness breaks more standard tricks...

A lot of thermal FT is generalising vacuum to work in medium!

Basic fermionic example

[Your favourite QFT textbook] will have something like:

$$\int_P \frac{d^D P}{(2\pi)^D}\left[P^2+m^2\right]^{-\alpha} \quad \langle D=d+1$$

$$=\left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)^{\frac{4-D}{2}} \frac{\Gamma(\alpha-D/2)}{(4\pi)^{D/2}\Gamma(\alpha)}\left[m^2\right]^{D/2-\alpha}$$

"one-loop master integral"

What is its equivalent in a medium?

$\sum_{n\in\mathbb{N}_+} n=-\frac{1}{12}$ has plenty of applications in thermal QFT :)

[assume $m \ll T$ or μ: the finite-m version has no closed form]

$$T\sum_{n\in\mathbb{Z}}\int_p \frac{d^d p}{(2\pi)^d}\left\{\left[\pi T(2n+1)-i\mu\right]^2+p^2\right\}^{-\alpha}$$

$$\equiv \SumInt_{\{P\}} \frac{1}{P^{2\alpha}}$$

Brackets = fermion

[We can use the vacuum result!]

$$=T\left(\frac{e^{\gamma_E}\Lambda^{2}}{4\pi}\right)^{\frac{4-d}{2}} \frac{\Gamma(\alpha-d/2)}{(4\pi)^{d/2}\Gamma(\alpha)}\sum_{n\in\mathbb{Z}}\left\{\left[\pi T(2n+1)-i\mu\right]^2\right\}^{d/2-\alpha}$$

$$=\left(\frac{e^{\gamma_E}\Lambda^{2}}{4\pi}\right)^{\frac{4-d}{2}} \frac{\Gamma(\alpha-d/2)}{(4\pi)^{d/2}\Gamma(\alpha)}\times(2\pi)^{d-2\alpha}\times T^{d-2\alpha+1}$$

$$\times\left[\zeta\left(\tfrac{1}{2}-i\mu/2\pi T\right)+\zeta\left(\tfrac{1}{2}+i\mu/2\pi T\right)\right]$$

Hurwitz Zeta

How would something like $\int_P \frac{P^\mu P^\nu}{P^2+m^2}$ change?

Quantum Field Pressure

Pressure ~ Free energy ~ Equation of State

Finnish proverb: "Rakkaalla lapsella on monta nimeä"

EoS is the $\boxed{\text{basic building block}}$ of NS modelling. How to get?

$$\Omega = \ln \int \mathcal{D}\phi \exp[-S(\phi)]$$

path integral

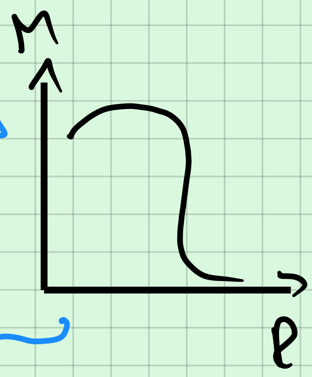
← action

Problem: e^{-S} complex @ finite μ
not a prob. measure → can't do Monte Carlo [sign problem]

Still, idea is there: This is how you get an EoS from microscopic theory for your favourite QFT:
Evaluate $p=p(\mu)$, or $p=p(\mu(n))$

[recall stat mech $n_f = \partial p / \partial \mu_f$]
if you prefer densities (good in nucl. mat.)

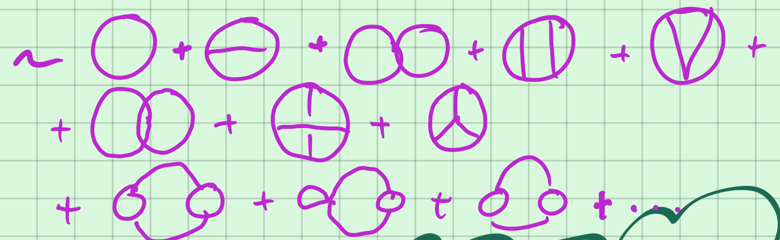
Fix action S
↳ Evaluate $\Omega \sim \ln \int \mathcal{D}\phi\, e^{-S}$
↳ Use as TOV eqn. input
↳ See if your results make sense



I like perturbation theory (weak coupling) for pressure estimates:

pressure ~ sum of bubble diagrams

$\Omega_{\text{el.+quark}} \sim$



In NS conditions, what can we assume about the scales?
Oooo

Nonstatic Quantities

Quiescent NSs are "easy"
but in 2026 we want more:
NS collisions are hot(-ish)
and dynamic near-equilibrium
processes

$\hookrightarrow$ need to understand
EoS at finite T (up to ~100 MeV)
and near-equilibrium physics like
transport phenomena
[cf. heavy-ion phys. $T \gg \mu$]

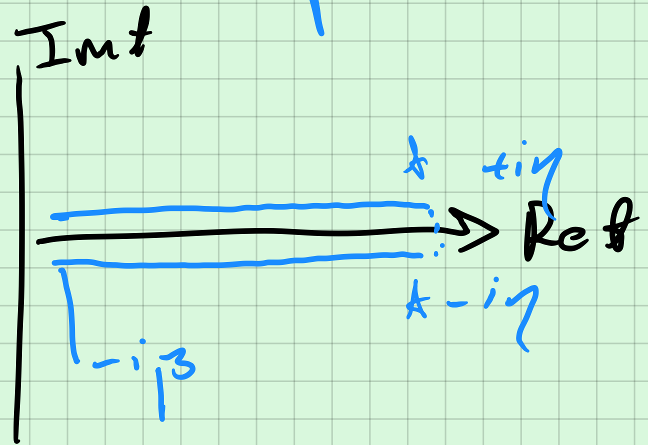
Still very little study at
finite μ.

Come talk to me about
bulk viscosity :P

Dynamic processes ~ t-dependence

$\longleftrightarrow$ can't cheat and use real x^0 to
encode thermal info
instead, need to double fields
"$\phi \mapsto (\phi_1, \phi_2)$"

and use complex time coordinate



Schwinger
-Keldysh
formalism

Eg propagator $G \mapsto \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$

"Thermalness" appears as distribution
functions in Feynman rules.

17

Free pressure from a QFT

$$S = \int_{S^1_\beta \times \mathbb{R}^3} dx\, \bar\psi [\not\partial - \mu \gamma_0] \psi \quad [\text{free Dirac fermion}]$$

$$\Omega_{DF} = \underbrace{-\ln \int \mathcal{D}\bar\psi\psi\, e^{-S}}_{\substack{\text{Quadratic action} \to \text{functional det.} \\ \text{(Grassman) Gaussian integral}}} = -\ln \operatorname{Det} \not\partial \overset{\text{mom. space}}{=} -2 \underset{\{Q\}}{\int\!\!\!\!\!\sum} \ln Q^2$$

Result must agree w/ stat. mech (free relativistic fermion gas!) but nice sanity check

up to terms without scale,

$$T \sum_n \int \ln\{[(2n+1)\pi T - i\mu]^2 + p^2\} \quad \frac{d}{d\beta} \rightsquigarrow n_F(p+\mu)$$

$$= \ldots = T \ln(1 + e^{-(p-\mu)\beta}) + \underbrace{T \ln(1 - e^{-(p+\mu)/T})}$$

Integrating over $p \in \mathbb{R}^d \simeq \mathbb{R}^3$,

$$\Omega_{DF} \overset{d=3}{=} -\frac{\pi^2}{45} \times \frac{7}{4} T^4 - \frac{1}{12\pi^2} (\mu^4 + 2\mu^2 T^2)$$

} Knowing how to do integrals is all well and good, but "big" calcs are hopeless without Mathematica

what's $\int_{\not p} \ln p^2$ used here?

$\Delta \equiv$ laplacian

Can do eg QCD $SU(N_c)$ gauge fields $+ N_f$ Dirac fermions just as easily)

$$\Omega^{LO}_{QCD} = -\ln \frac{\operatorname{Det} \not\partial^{N_f N_c} \operatorname{Det}(-\Delta)^{d_A}}{\sqrt{\operatorname{Det}(-\delta^\beta_\alpha \Delta + (1 - \frac{1}{\xi}) \partial^\alpha \partial_\beta)^{d_A}}}$$

$$\overset{d=3}{=} -\frac{\pi^2}{45}(d_A + N_c N_f) T^4 - \frac{1}{12} N_c \sum_{i=1}^{N_f} (\mu_i^2 + 2T^2) \mu_i^2$$

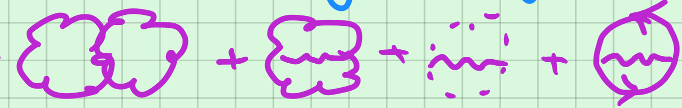
[cf. "Bag models" of QCD: this w/ some model constants]

I use pQCD as an example, but the idea is the same w/ any nuclear EFT or NJL, etc.

NLO Pressure

Start by drawing diags

$$\Omega^{QCD}_{NLO} =$$



[Very easy to contract w/ FeynCalc or similar]

$$= \frac{g_s^2}{4} d_A (d-1) \Big\{ N_c (d-1) \Big(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \Big)^2$$

$$- \frac{2^{\lfloor (d+1)/2 \rfloor}}{2} \sum_{i=1} \Big[2 \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2} - \underbrace{\Big(\sum\!\!\!\!\!\!\int_{\{Q\}} \frac{1}{Q^2} \Big)^2}_{} \Big] \Big\}$$

only part for $T=0$!

$$\overset{d=3}{=} \frac{g_s^2}{144} d_A \Big[\Big(N_c + \frac{5}{4} N_f \Big) T^4 + \frac{9}{4\pi^4} \sum_{i=1}^{N_f} (\mu_i^2 + 2\pi^2 T^2) \mu_i^2 \Big]$$

So far everything is finite!

Eventually, $1/\varepsilon$'s appear like in vacuum QFTs: Contributions from high energies diverge

$1/\varepsilon$'s cancel against renormalization of the couplings, masses...: Here

$$g_B^2 \mapsto g_R^2(\bar{\Lambda}) \Big[1 + \frac{\beta_0}{\varepsilon} g_R^2(\bar{\Lambda}) \Big]$$

$$g_B^2 \, c_{0,1} + g_B^4 \Big(\frac{c_{-1,2}}{\varepsilon} + c_{0,2} \Big) + \dots$$

$$\mapsto g_R^2(\bar{\Lambda}) c_{0,1} + g_R^4(\bar{\Lambda}) c_{0,2} + g_R^4(\bar{\Lambda}) \big[c_{0,1}\beta_0 + c_{-1,2} \big] / \varepsilon$$

[= 0 if you did things correctly]

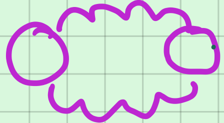
But this only works for UV!

In a medium, IR divergences show up!

Intuitively, why are IR divergences a problem in a medium? (think finite T)

Hard Thermal Loops of Braaten & Pisarski

Keep going w/ Ω:

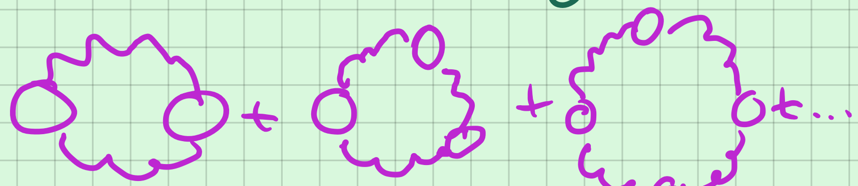
this diag  is a problem

Can you see why just by looking at it?

It leaves a $1/\varepsilon$ that is <u>not</u> cancelled by $g_B \mapsto g_R$

Why no  ?

Solution: resum



 ← LO free energy of Hard Thermal Loop

$\therefore$ In a medium, IR sensitive diagrams need <u>resumming</u>

We can compute this (see next slide for how to achieve this without much headache)

$$\Omega^{HTL}_{LO} = -\frac{1}{2}\sum_{n\in\mathbb{N}_+}\frac{(-1)^n}{n}\int_P \mathrm{Tr}\left[\Pi(P)\,G(P)\right]^n$$

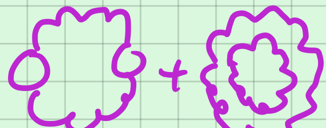
[Two tensor components T, L] →

$$= \frac{d_A}{2}\int_P\left[(d-1)\ln\left(1+\frac{\Pi_T(P)}{P^2}\right)+\ln\left(1+\frac{\Pi_L(P)}{P^2}\right)\right]$$

$$\approx -\frac{d_A}{(8\pi)^2}m_E^2\left(\frac{m_E}{\Lambda}\right)^{-2\varepsilon}\left[\frac{1}{2\varepsilon}+1.17201+O(\varepsilon)\right]$$

↑ Effective mass $\propto g^2\mu^2$

HTLs have a scale m_E, normal diags have a scale μ. Combining terms leaves a log

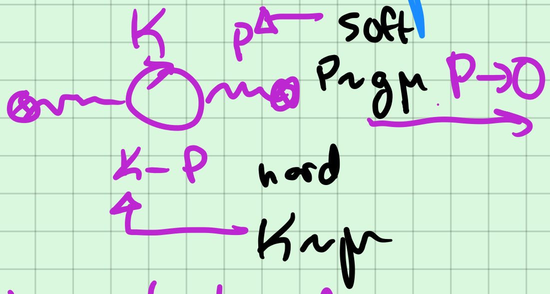
 $= -\frac{d_A m_E^4}{(8\pi)^2}\left[\left(\frac{m_E}{\Lambda}\right)^{-2\varepsilon}\frac{1}{2\varepsilon}-\left(\frac{\mu}{\Lambda}\right)^{-2\varepsilon}\frac{1}{2\varepsilon}\right]+O(1)$

$$\boxed{\simeq -\frac{d_A m_E^4}{(8\pi)^2}\ln\frac{m_E}{\mu}.}$$

Only talking about the IR sector at large μ here: Different tricks at large T

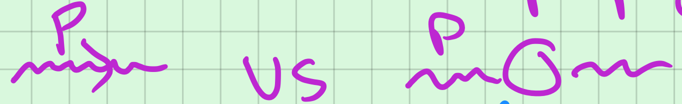
More on HTLs

2nd PoV: Compute self energy



$\xrightarrow{P \to 0} \# g^2 \mu^2 + \mathcal{O}\left(\frac{P^2}{\mu^2}\right)$

Now look at a propagator



with $P \sim g\mu$ soft

#2 should be suppressed, it's NOT:

$$\frac{1}{P^2} \sim \frac{1}{g^2\mu^2} \quad \text{vs} \quad \frac{1}{P^2}\, g^2\mu^2\, \frac{1}{P^2} \sim \frac{1}{g^2\mu^2}!$$

aribitrarily many insertions needed:

$$\text{〰} = \text{〰} + \text{〰○〰} + \text{〰○〰○〰}$$

$$\overset{d=3}{=} \dots = \frac{\delta_{\mu i}\delta_{\nu j}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)}{P^2 + m_E^2 \frac{P^2}{2p^2}\left[\frac{p_0^2}{P^2} - i\frac{p_0}{p}\ln\left(\frac{ip_0+p}{ip_0-p}\right)\right]} + \frac{\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} - \delta_{\mu i}\delta_{\nu j}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)}{P^2 + m_E^2 \frac{Q^2}{q^2}\left[1 - \frac{ip_0}{2p}\ln\left(\frac{ip_0+p}{ip_0-p}\right)\right]}$$

Introduces new effective mass

$$m_E^2 \overset{d=3}{=} g_s^2\left[N_f \frac{\mu^2}{2\pi^2} + (2N_c + N_f)\frac{T^2}{6}\right]$$

What is the physics behind m_E^2?

Can play a similar game with 3, 4, …-point functions

$$\text{〰} \;\Gamma \longrightarrow \Gamma_0 + \underbrace{\delta\Gamma}_{\propto m^2}$$

No closed form

3rd PoV: HTL as an EFT

Can also derive standard HTL from Lagrangian (just gluon here)

$$\mathcal{L}_{H} = \frac{m_D^2}{2}\,\mathrm{Tr}\int_{v \in S^2} F^{\mu\alpha}\frac{V_\alpha V_\beta}{(V\cdot D)^2}F^{\beta}{}_{\mu}$$

fin.

Thank you!

Enjoy Barcelona,

CS@CD 2026,

AND ASK QUESTIONS

(Whether it's
now/later
from me/others
about physics/
what to do in Barcelona
or whatever)

Bonus

Recent Developments (selfish pQCD PoV)

B1

↑ likely not dissimilar to Aleksi's plenary's pQCD parts...

[everything until now was methodology: applicable in general]

1) For cold NSs, pressure computed to 3 loops in '77 (!)

BIIIIG gap while EFTs were being developed

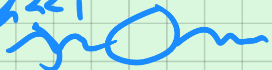
Recent push to $\mathcal{O}(g_s^6)$ w/ HTLs:

$\mathcal{O}(g^6 \ln^2 g)$ in '18 (by me & others)

$\mathcal{O}(g^6 \ln g)$ in '23

$\mathcal{O}(g^6)$ VERY SOON :)

Needed to understand HTL past Braaten & Pisarski: What kind of corrections to

$K \ll T$  past $\mathcal{O}(g^2)$

↳ Rapid recent progress

& needed new tricks for computing numerics.

Finite T & μ progress progress more consistent. Recently figured out finite m, μ, T

↳ Could calculate Bulk viscosity

$\zeta \propto m_s^4$

(w/out m_s to break chiral symmetry, matter is conformal $\Longrightarrow$ ideal.

Turns out: Relativistic dense matter is like leggings :)

Lots of other interesting approaches for large μ, holography option for studying ~nearby theories, Functional Renormalisation Group has found out interesting things about Conensates and c_s^2

And I'm sure you'll hear lots about nuclear phase from ppl who know it much, much better than I.

Bonus: Quick derivation of ITF

$\langle \mathcal{O} \rangle_{\substack{\text{stat}\\\text{mech}}} = Z^{-1}\,\mathrm{Tr}\,\mathcal{O} e^{-\beta H}$, $Z = \mathrm{Tr}\, e^{-\beta H}$

vs (Eucl.)

path integral ~ trace in a large func. space (in the Gelfand triplet sense blah blah blah)

$\langle \mathcal{O} \rangle_{\substack{QFT,\\T=0}} = Z^{-1} \int \mathcal{D}\phi\, \mathcal{O} e^{-\int d^D x \mathcal{L}_E} \equiv Z^{-1}\, \underbrace{\mathrm{Tr}}_{}\left[\mathcal{O} e^{-S_E}\right]$, $Z = \int \mathcal{D}\phi\, e^{-S_E}$

Take states $|\phi_i\rangle$, $|\phi_f\rangle$ in Mink.

Evaluate @ time t and analytically continue:

$\langle \phi_f | e^{-iHt} | \phi_i \rangle = \int_{\phi(0)=\phi_i}^{\phi(t)=\phi_f} \mathcal{D}\phi\, e^{iS} \xmapsto{AC} \langle \phi_f | e^{-\beta H} | \phi_i \rangle = \int_{\phi(t)=\phi_i}^{\phi(t-i\beta)=\pm\phi_f} \mathcal{D}\phi\, e^{-S_E}$

$\mathrm{Tr}\, e^{-\beta H} = \int d\phi_i \int_{\phi(t)=\phi_i}^{\phi(t-i\beta)=\pm\phi_i} \mathcal{D}\phi\, e^{-S_E}$ } tracelike ↪ BC.s must be periodic/antiperiodic

Requiring $\langle id \rangle = 1$,
⟶ normalisation

periodic/antiperiodic for static

KMS $\langle \mathcal{O}_1^M(t_1, x_1)\, \mathcal{O}_2^M(t_2, x_2) \rangle = Z^{-1}\, \mathrm{Tr}\left[\mathcal{O}_1^M(t_1,x_1)\, \mathcal{O}_2^M(t_2,x_2)\, e^{-\beta H}\right]$

$= Z^{-1}\, \mathrm{Tr}\left[\mathcal{O}_1^M(t_1,x_1)\, e^{-\beta H}\, \mathcal{O}_2(t_2 - \beta i, x_2)\right] = \langle \mathcal{O}_2^M(t_2 - i\beta, x_2), \mathcal{O}_1^M(t_1, x_1) \rangle$

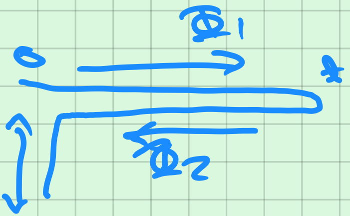
Bonus: Briefly on RTF

In RTF, instead

$$\langle O(t)\rangle = \frac{1}{Z}\int d\phi\, d\phi'\, d\phi''\, d\phi''' \int_{\phi_E(0)=\phi'''}^{\phi_E(-i\beta)=\pm\phi''} \mathcal{D}\phi_E \int_{\phi_1(0)=\phi''}^{\phi_1(t)=\phi'} \mathcal{D}\phi_1 \int_{\phi_2(0)=\phi'''}^{\phi_2(t)=\phi} \mathcal{D}\phi_2\, \langle\phi| O(t) |\phi'\rangle\, e^{iS[\phi_1]-iS[\phi_2]-S_E[\phi_E]}$$

match

fields on a contour



$\phi_2 \sim \phi_E$

Euser basis: $(\phi_r, \phi_a) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} (\phi_1, \phi_2)$

example: gluon propagator

$$G^{ab}_{\mu\nu}(k,p) = \left[g_{\mu\nu} \begin{pmatrix} D_{rr} & D_R \\ D_A & 0 \end{pmatrix} + (1-\xi) P_{\mu\nu} \begin{pmatrix} \frac{-D_{rr}^2}{(\frac{1}{2}+n_B(p_0))} & D_R^2 \\ D_A^2 & 0 \end{pmatrix} \right]$$

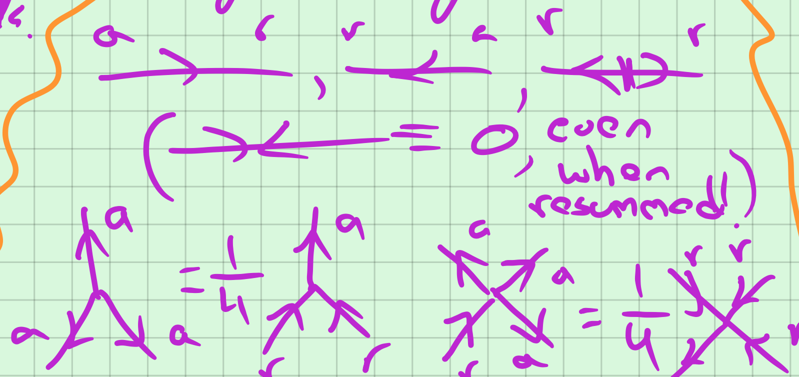
$$D_{A/R} = \frac{-i}{p^2 \pm i\eta p^0}, \quad D_{rr} = \left(\frac{1}{2}+n_B(p_0)\right) 2\pi\delta(p^2)\,\mathrm{sgn}(p_0)$$

Normally, every diag doubles w/ type 1, 2 fields: this simplifies: Add "causal labelling"

$a \longrightarrow r \qquad r \longleftarrow a$

(arrows toward r)

Only Core objects



(Only for bare quantities!)

Simplifying consequence:

Closed Causal Loops vanish BARE

rr-propagator contains "thermalness"

aa-propagator vanishes identically (all orders!)