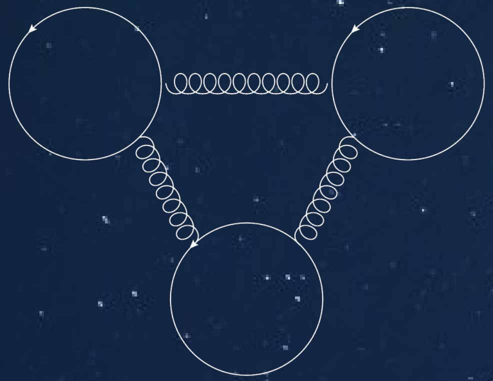
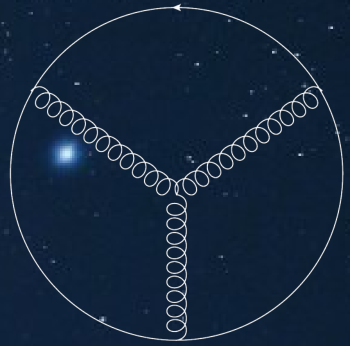
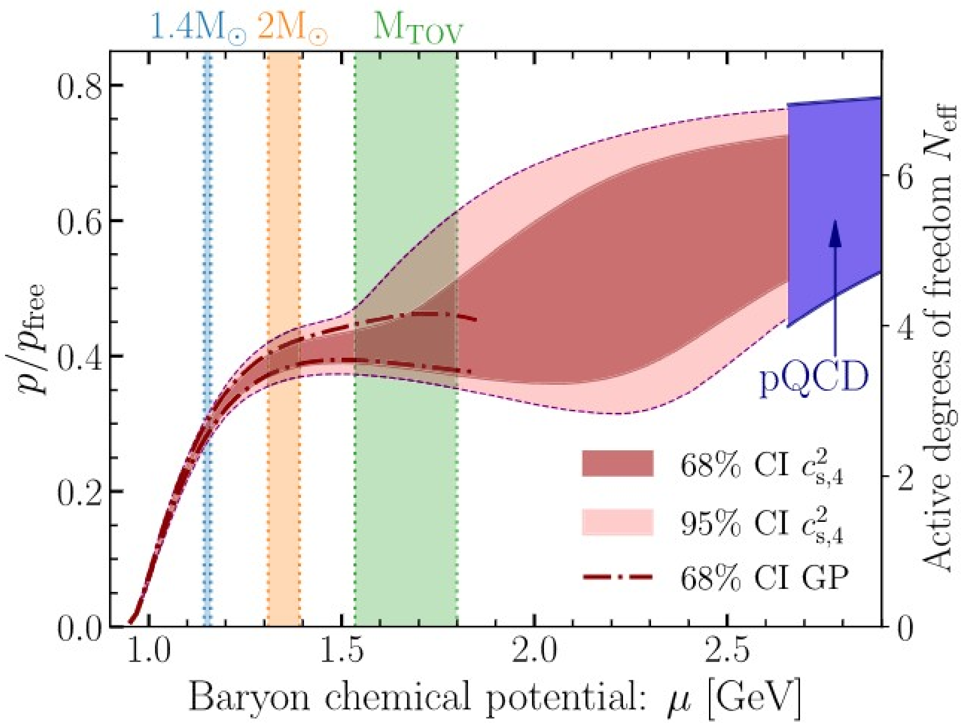
Cold quark matter from multi-loop techniques





Max Delto, CERN
in collaboration with Samuel Abreu (CERN),
Aleksi Kurkela (University of Stavanger)

Quark matter in Neutron star cores

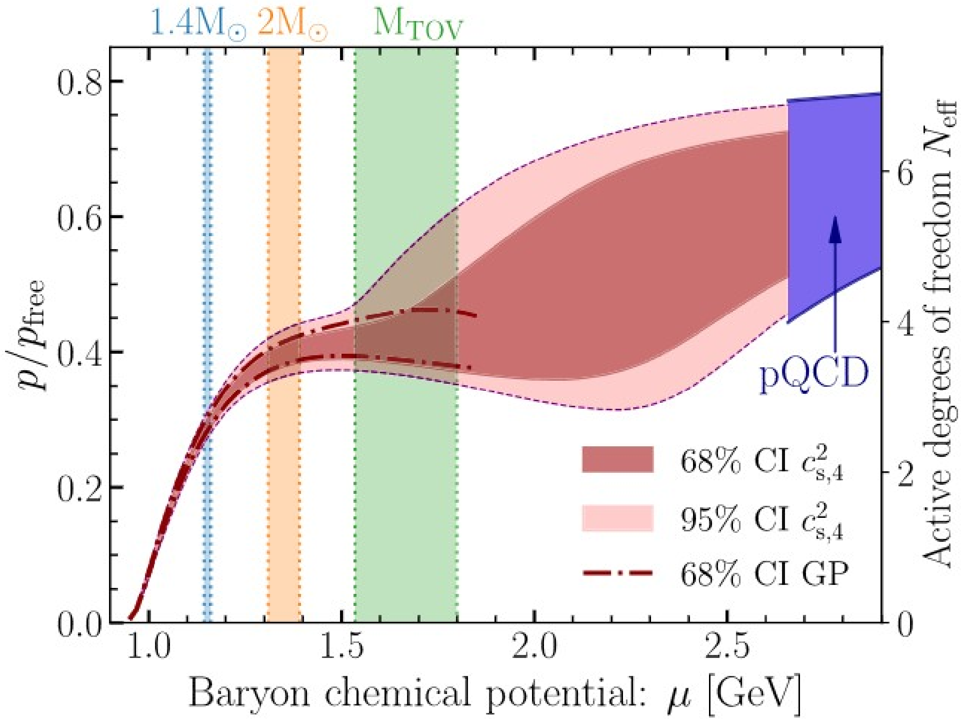


[Annala et al. PRL 120 (2018);
Nature Phys. 16 (2020); PRX 12 (2022);
Nature Comm. 14 (2023)]

- high- and low-density limits under theoretical control
- systematic use of these limits + observational data

"in the cores of maximally massive stars, the EoS is consistent with quark matter"

Quark matter in Neutron star cores



[Annala et al. PRL 120 (2018);
Nature Phys. 16 (2020); PRX 12 (2022);
Nature Comm. 14 (2023)]

<u>This talk:</u> techniques to obtain more precise pQCD predictions.

Cold quark matter

$$p = -\Omega = -\ln \int \mathcal{D}\Psi \mathcal{D}\overline{\Psi} \mathcal{D}A \, \exp(-S_{\mathrm{QCD}}(\mu))$$

pQCD

$$p = p_{\mathrm{FD}} + \alpha_s p_1^h + \alpha_s^2 p_2^h + \alpha_s^3 p_3^h$$
$$+ \alpha_s^2 p_2^s + \alpha_s^3 p_3^s$$
$$+ \alpha_s^3 p_3^m$$

mixed modes

soft modes

only partial results for hard N3LO piece

Selected for a Viewpoint in *Physics*
PHYSICAL REVIEW D **81,** 105021 (2010)

Cold quark matter

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(Received 10 February 2010; published 24 May 2010)

We perform an $\mathcal{O}(\alpha_s^2)$ perturbative calculation of the equation of state of cold but dense QCD matter with two massless and one massive quark flavor, finding that perturbation theory converges reasonably well for quark chemical potentials above 1 GeV. Using a running coupling constant and strange quark mass, and allowing for further nonperturbative effects, our results point to a narrow range where absolutely stable strange quark matter may exist. Absent stable strange quark matter, our findings suggest that quark matter in (slowly rotating) compact star cores becomes confined to hadrons only slightly above the density of atomic nuclei. Finally, we show that equations of state including quark matter lead to hybrid star masses up to $M \sim 2M_\odot$, in agreement with current observations. For strange stars, we find maximal masses of $M \sim 2.75M_\odot$ and conclude that confirmed observations of compact stars with $M > 2M_\odot$ would strongly favor the existence of stable strange quark matter.

DOI: 10.1103/PhysRevD.81.105021 PACS numbers: 12.38.−t, 12.39.−x, 21.65.Qr, 26.60.−c

PHYSICAL REVIEW LETTERS **127,** 162003 (2021)

Soft Interactions in Cold Quark Matter

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(Received 26 March 2021; revised 12 July 2021; accepted 25 August 2021; published 15 October 2021)

Accurate knowledge of the thermodynamic properties of zero-temperature, high-density quark matter plays an integral role in attempts to constrain the behavior of the dense QCD matter found inside neutron-star cores, irrespective of the phase realized inside the stars. In this Letter, we consider the weak-coupling expansion of the dense QCD equation of state and compute the next-to-next-to-next-to-leading-order contribution arising from the non-Abelian interactions among long-wavelength, dynamically screened gluonic fields. Accounting for these interactions requires an all-loop resummation, which can be performed using hard-thermal-loop (HTL) kinematic approximations. Concretely, we perform a full two-loop computation using the HTL effective theory, valid for the long-wavelength, or soft, modes. We find that the soft sector is well behaved within cold quark matter, contrary to the case encountered at high temperatures, and find that the new contribution decreases the renormalization-scale dependence of the equation of state at high density.

DOI: 10.1103/PhysRevLett.127.162003

PHYSICAL REVIEW D **68**, 054017 (2003)

Pressure of QCD at finite temperatures and chemical potentials

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(Received 4 June 2003; published 23 September 2003)

We compute the perturbative expansion of the pressure of hot QCD to order $g^6 \ln g$ in the presence of finite quark chemical potentials. In this process we evaluate all two- and three-loop vacuum diagrams of the theory at arbitrary T and μ and then use these results to analytically verify the outcome of an old order g^4 calculation of Freedman and McLerran for the zero-temperature pressure. The results for the pressure and the different quark number susceptibilities at high T are compared with recent lattice simulations showing excellent agreement especially for the chemical potential dependent part of the pressure.

DOI: 10.1103/PhysRevD.68.054017 PACS number(s): 12.38.Mh, 11.10.Wx

PHYSICAL REVIEW D VOLUME 16, NUMBER 4 15 AUGUST 1977

Fermions and gauge vector mesons at finite temperature and density. III. The ground-state energy of a relativistic quark gas*

Barry A. Freedman and Larry D. McLerran
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(Received 12 November 1976)

We calculate the ground-state energy of a quark gas up to and including effects of fourth order in the quark-gluon coupling. Renormalization-group techniques are used to rewrite the results of the fourth-order calculation in terms of a chemical-potential-dependent coupling. The chemical-potential-dependent coupling approaches zero at high densities. We argue that at hadronic matter densities a phase transition between quark matter and hadronic matter occurs. We show the existence of this phase transition for a three-color, one-flavor quark gas. We find a close correspondence between the results of quantum chromodynamics at large coupling with no bag constant and quantum chromodynamics at small coupling with a bag constant.

HIP-2025-01/TH

Quark matter at four loops: hardships and how to overcome them

Aapeli Kärkkäinen,[1] Pablo Navarrete,[1] Mika Nurmela,[1] Risto Paatelainen,[1] Kaapo Seppänen,[1] and Aleksi Vuorinen[1]
[1]*Department of Physics and Helsinki Institute of Physics, P.O. Box 64, FI-00014 University of Helsinki, Finland*

Knowledge of the pressure of cold and dense quark matter (QM) is known to significantly constrain the equation of state of neutron-star matter, a quantity playing a key role in deciphering the stars' internal structure. In this work, we make important progress towards determining the last unknown contribution to the pressure at Next-to-Next-to-Next-to-Leading Order (N3LO) in the strong coupling constant α_s, available through the sum of all four-loop vacuum diagrams of dense Quantum Chromodynamics. We demonstrate the cancellation of both the covariant gauge parameter and all infrared (IR) divergences in the sum, showcasing the effective-field-theory paradigm in action. For the remaining IR-finite four-loop integrals, we demonstrate the efficacy of the dense Loop Tree Duality method — a novel numerical framework for multiloop calculations in thermal field theory. Together, our results show that completing the N3LO pressure of cold QM is no longer merely possible but for the first time within reach.

+ [.....]

Cold quark matter

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pQCD

$$p = p_{\mathrm{FD}} + \alpha_s p_1^h + \alpha_s^2 p_2^h + \alpha_s^3 p_3^h$$
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mixed modes

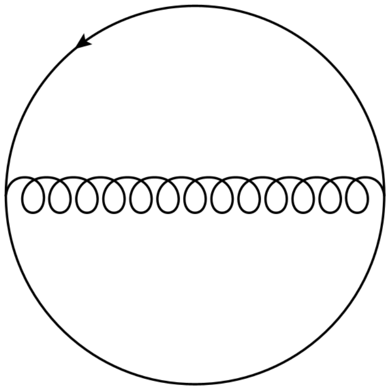
soft modes

only partial results for hard N3LO piece

Can we make use of (semi-)analytic techniques developed in the context of vacuum Feynman Integrals; especially given the low number of scales, i.e. μ (and m)?

Dense QFT calculations

$$\widetilde{R} = R + i\mu(1,\mathbf{0})$$

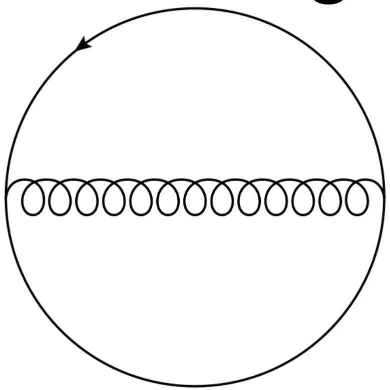


$$= \int \frac{\mathrm{d}^D p \; \mathrm{d}^D q}{\left[\widetilde{P}^2 + m^2\right]\left[\widetilde{Q}^2 + m^2\right]\left[(P-Q)^2\right]}$$

- Diagram generation (e.g. *QGRAF*)
- Integrand symmetrization (e.g. *Feynson*)
- numerator algrabra (e.g. *FORM*)

Cutting rules [Ghisoiu et al., Nucl.Phys.B 915 (2017)]

$$\widetilde{R} = R + i\mu(1,\mathbf{0})$$

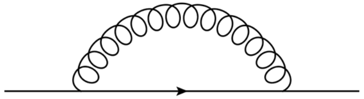


$$= \int \frac{\mathrm{d}^D p\, \mathrm{d}^D q}{\big[\widetilde{P}^2 + m^2\big]\big[\widetilde{Q}^2 + m^2\big]\big[(P-Q)^2\big]}$$

- organizes residue calculus
- yields vac. propagators (w/o μ)
- D-dim regularisation for UV & IR
- no Lorentz inv., but spatial rotational inv.

$$\sim \frac{\mathrm{d}^D q}{\big[Q^2 + m^2\big]\big[(Q-P)^2\big]}$$

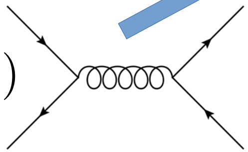
$$= 2\int \frac{\mathrm{d}^{D-1}\boldsymbol{p}}{2E_{\boldsymbol{p}}}\theta(\mu - E_{\boldsymbol{p}})$$



$$\Big|_{p_0 \to iE_{\boldsymbol{p}}} \qquad \big[\equiv I^{\theta}\big]$$

$$\sim \frac{1}{\big[(P-Q)^2\big]}$$

$$+ \int \frac{\mathrm{d}^{D-1}\boldsymbol{p}}{2E_{\boldsymbol{p}}}\frac{\mathrm{d}^{D-1}\boldsymbol{q}}{2E_{\boldsymbol{q}}}\theta(\mu - E_{\boldsymbol{p}})\theta(\mu - E_{\boldsymbol{q}})$$



$$\Big|_{p_0 \to iE_{\boldsymbol{p}},\, q_0 \to iE_{\boldsymbol{q}}} \qquad \big[\equiv I^{\theta\theta}\big]$$

...towards Feynman integrals

- restore Lorentz covariance

$$\frac{\mathrm{d}^{D-1}\boldsymbol{p}}{2E_{\boldsymbol{p}}}\theta(\mu - E_{\boldsymbol{p}}) \;=\; \mathrm{d}^D p\,\delta(p^2 - m^2)\,\theta(\mu - n.p)$$

- Interpret on-shell conditions / discontinuity as propagator, commonly known as reverse unitarity [Anastasiou and Melnikov, Nucl. Phys. B 646 (2002)]

$$(2\pi i)\,\delta^+(q^2 - m^2) = \lim_{\sigma\to 0^+}\left[\frac{1}{q^2 - m^2 + i\sigma} - \frac{1}{q^2 - m^2 - i\sigma}\right] \equiv \frac{1}{[q^2 - m^2]_{\delta+}}$$

- arrive at vacuum FIs (over constrained volume), e.g.

$$I^{\theta\theta} = \int \frac{\mathrm{d}^D p\,\mathrm{d}^D q\,\theta(\mu - p.n)\,\theta(\mu - q.n)}{\left[p^2 - m^2\right]_{\delta+}\left[q^2 - m^2\right]_{\delta+}\left[(p-q)^2\right]}$$

Integration-by-parts identities

integrals over gradients vanish in dim. reg. [Chetyrkin and Tkachov, Nucl.Phys.B 192 (1981)]

also useful for FIs over constrained domains [Baranowski,MD,Melnikov,Wang, JHEP 02 (2022)]

$$
\begin{aligned}
0 &= \int \mathrm{d}^D k \, (\partial_k \cdot v) \, \theta[f(k)] \frac{\mathcal{N}(k)}{\mathcal{D}(k)} \\
&= \int \mathrm{d}^D k \, \theta[f(k)] \, (\partial_k \cdot v) \, \frac{\mathcal{N}(k)}{\mathcal{D}(k)} + \int \mathrm{d}^D k \, \frac{\mathcal{N}(k)}{\mathcal{D}(k)} \, \delta[f(k)](\partial_k \cdot v) \, f(k)
\end{aligned}
$$

- simple (but powerful) relation to generate linear relations between integrals with different numerator and denominator powers.
- ``IBP reduction'': express any integral through sum of few, simple ``master integrals'' fully automatized in many codes for <u>standard FIs</u> (i.e. no θ)
- in practice, we generate large linear systems [Laporta, Int.J.Mod.Phys.A 15 (2000)] and solve them using *Ratracer* [Magerya, 2211.03572]
- in our case, trade numerator/denominator powers, as well as θ's with δ's

Differential equations for MI

[Kotikov, Phys.Lett.B 254 (1991)]

- Integrals close under IBPs → satisfy linear PDE system in kinematic variables

$$\frac{\partial}{\partial x}\vec{I} \overset{\text{take derivatives}}{=} \ldots \underset{\text{IBP reduction}}{=} A_x(\epsilon,\mu,m^2)\,\vec{I} \qquad x\in\{\mu,m^2\}$$

- vast literature on how to simplify DEs, handle appearing special functions, solve numerically,…[...]

- effectively turned ``computing integrals'' into ``solving DEs'' + ``finding boundary conditions'', the latter step can be simplified even further

Differential equations for MI: auxiliary mass flow

[Liu,Ma Comput.Phys.Commun. 283 (2023)]

- basic idea: insert auxiliary mass η into (subset of) propagators,
  use large η limit to impose boundary conditions

- has been fully automatized for standard FIs (*AMFlow*)
  (we derive DEs and BCs but make use of it's internal DE solver)

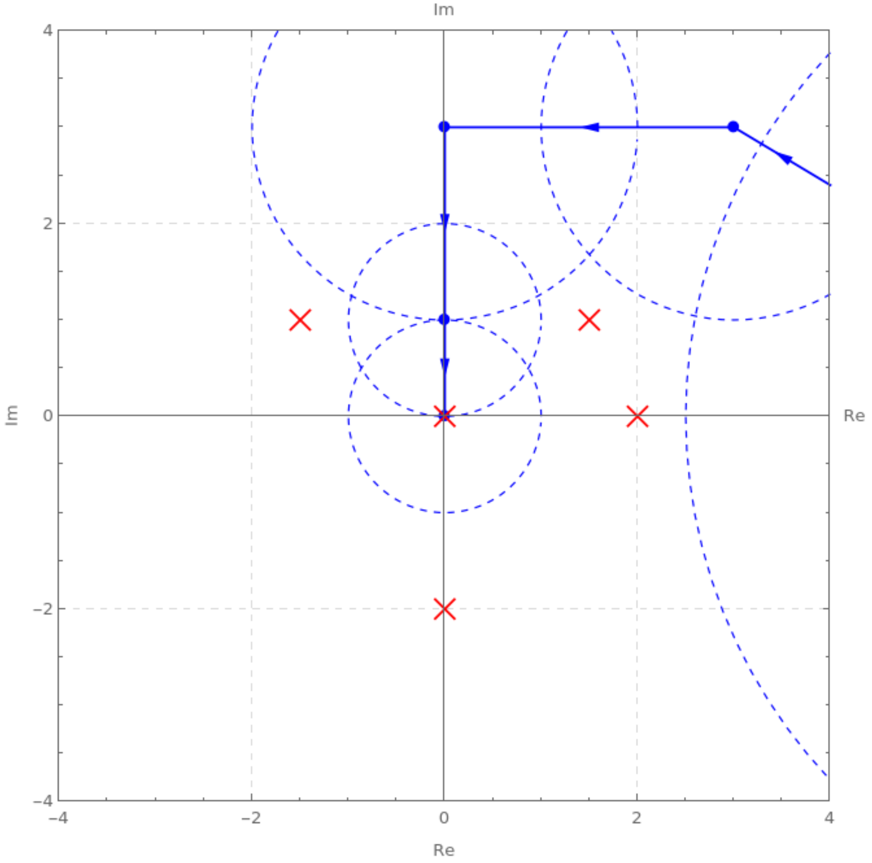
- for our example, BCs simplify:

$$J^{\theta\theta}(\eta) = \int \frac{\mathrm{d}^D p\, \mathrm{d}^D q\, \theta_p\, \theta_q\, \delta(p^2-m^2)\, \delta(q^2-m^2)}{\left[(p-q)^2+\eta\right]} = \frac{1}{\eta}\left[\int \mathrm{d}^D p\, \theta_p \delta(p^2-m^2)\right]^2 + \mathcal{O}\left(\frac{1}{\eta^2}\right)$$

$$J^{\theta}(\eta) = \int \frac{\mathrm{d}^D p\, \mathrm{d}^D q\, \theta_p \delta(p^2-m^2)}{\left[q^2-m^2+\eta\right]\left[(p-q)^2+\eta\right]} = \left[\int \mathrm{d}^D p\, \theta_p \delta(p^2-m^2)\right] \times \left[\int \frac{\mathrm{d}^D q}{\left[q^2+\eta\right]^2}\right] + \mathcal{O}\left(\frac{1}{\eta}\right)$$

Differential equations for MI: auxiliary mass flow

[Liu,Ma Comput.Phys.Commun. 283 (2023)]



- transport from BC at large η to η=0
  via successive series expansions

- final asymptotic series at η=0 take a form

$$J(\eta \approx 0) = \sum_{a \geq 0, b, c \geq 0} c_{abc}(\epsilon) \eta^{a+b\epsilon} \ln^c(\eta)$$

- recover original integrals from η=0 at finite ε

$$I = J(0) = c_{0,0,0}$$

- integrals with massive quarks require two step evolution

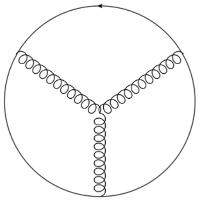
$$\{\eta = \infty, m^2\} \xrightarrow{\partial_\eta} \{\eta = 0, m^2 = 1/2\} \xrightarrow{\partial_{m^2}} \{m^2 = 0, 1\}$$

… in practice

$$\Bigg\{ \underbrace{\int \frac{\mathrm{d}^D\, q}{q^2+1}\ ,\ \int \frac{\mathrm{d}^{D-1}\, \boldsymbol{p}}{2E_{\boldsymbol{p}}}\theta(\mu - E_{\boldsymbol{p}})}_{\text{few inputs}}\ ,\ \text{IBP reduction}\ ,\ \text{(asymptotic) series expansions of DEQ} \Bigg\}$$

(high-precision evaluation allows reconstruction of analytic rational numbers)

$$p_2^h\Big|_{n_l^1} = \ \ldots\ +$$



$$= -\frac{n_l(N_c^2-1)}{4\pi^2}\left[\frac{17}{4N_c} + N_c\left(\frac{415}{36} + \frac{22}{3}\ln\left(\frac{\bar{\Lambda}}{2}\right)\right)\right]$$

Conclusions

- NS EoS considerably constrained by pQCD results
- Although high-order QFT calculations at nonzero density are cumbersome ...
- ...they can be linked to vacuum integrals, opening the door to a vast landscape of tools, such as (thermal/dense)LTD, integration-by-parts and differential equation method
- LTD/MC integration & (semi-)analytic methods complement each other
- stay tuned for upcoming N3LO results!