Bayesian Inference of Dense Matter Equation of State from Future High-Precision Neutron Star Radius Measurements

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Formerly known as Texas A&M University-Commerce (2024-1996), East Texas State University/College (1996-1957), East Texas State/Normal College (1957-1889)

Collaborators: B.J. Cai, X. Grundler, W.J. Xie and N.B. Zhang

Key issues:

1. **Can we learn anything new from high-precision R-data?**

- Hadronic EOS
- Hadron-quark transition and quark matter

2. How do we understand the role of observational data precision in inference?

- Analogy with risk aversion in economics and noise-induced drift in nonlinear dynamics
- Evidence for Jensen bias: **nonlinear mapping + varying uncertainty in data = systematic inference shift**

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What is the EOS?

- **Relation among P, ε, T, δ, μ**
- **Symmetry energy is a key uncertainty**

Empirical parabolic law of the EOS of cold, neutron-rich nucleonic matter

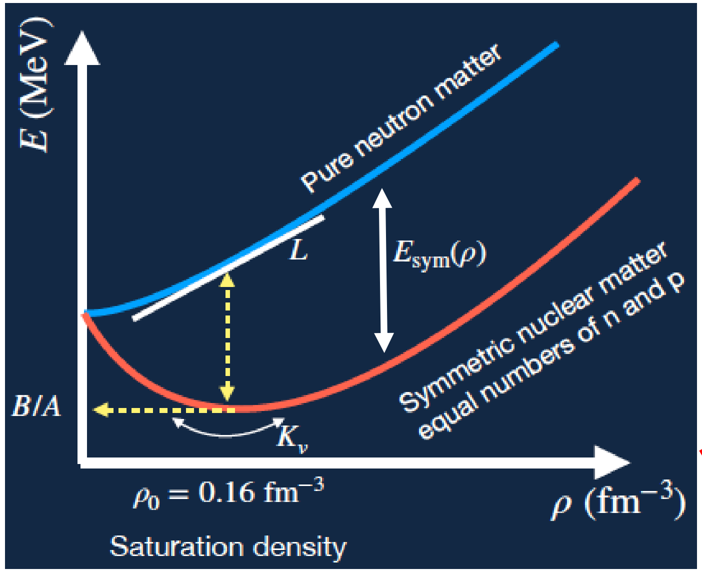
symmetry energy

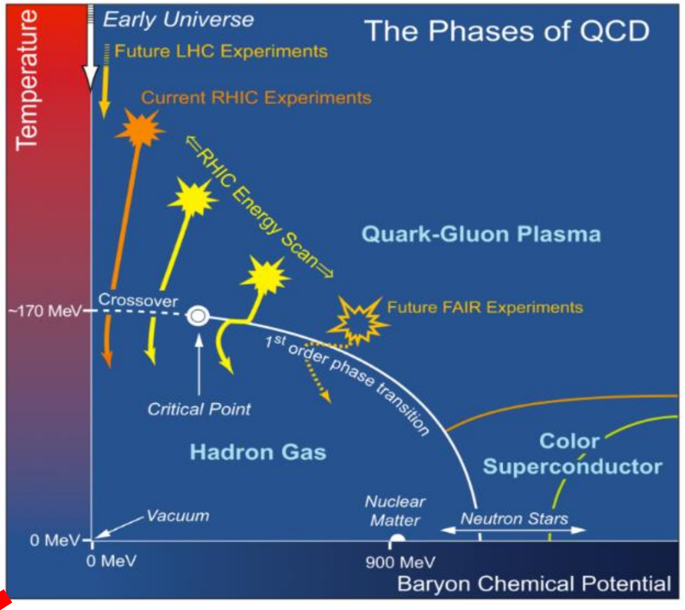
Isospin asymmetry δ

$$E(\rho_n,\rho_p) = E_0(\rho_n = \rho_p) + E_{sym}(\rho)\left(\frac{\rho_n - \rho_p}{\rho}\right)^2 + o(\delta^4)$$

Energy per nucleon in symmetric matter

Energy in asymmetric nucleonic matter





New opportunities

Isospin chemical potential

$\mu_I = E_{sym}(\rho) \cdot \delta$ in n-rich matter

Structures and collisions of heavy nuclei

Structures and mergers of neutron stars

Empirical parameterizations useful for meta-modeling (template) of EOS

incompressibility

$$E_0(\rho) = E_0(\rho_0) + \frac{K_0}{2}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^2 + \frac{J_0}{6}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^3 + \frac{Z_0}{24}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^4,$$

Relatively well-known

Poorly known

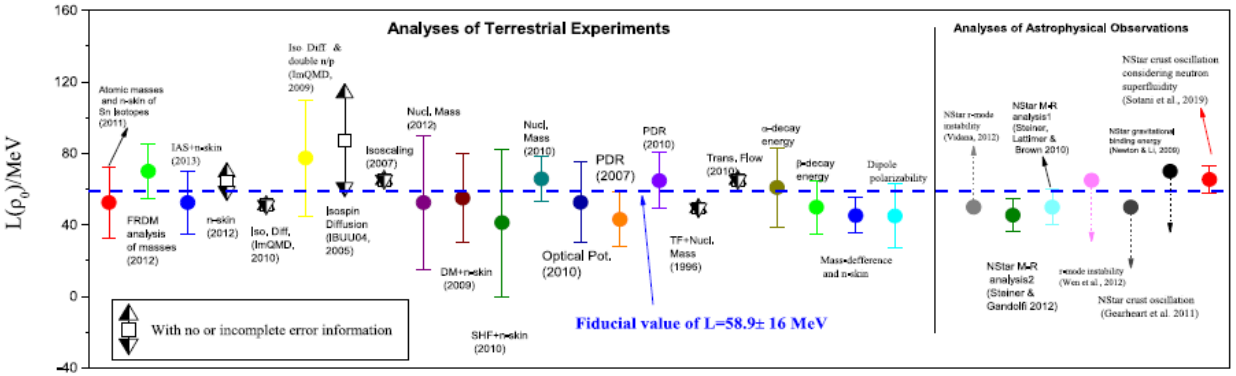
$$E_{sym}(\rho) = E_{sym}(\rho_0) + \frac{L}{3}\left(\frac{\rho}{\rho_0}-1\right) + \frac{K_{sym}}{18}\left(\frac{\rho}{\rho_0}-1\right)^2 + \frac{J_{sym}}{162}\left(\frac{\rho}{\rho_0}-1\right)^3 + \mathcal{O}\left[\left(\frac{\rho}{\rho_0}-1\right)^4\right]$$

slope curvature skewness kurtosis

- Expansion near saturation density
- Uncertainties grow at high density

Prior on L: Broad theoretical range L ≈ 60 ± 30 MeV with a few out-standing points

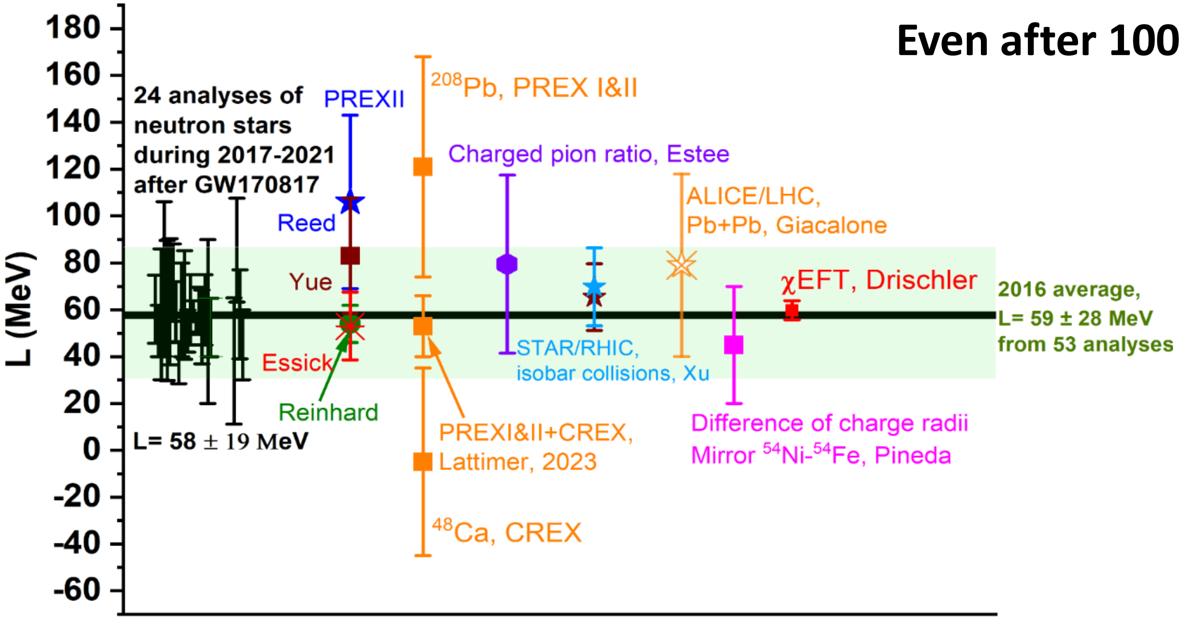
As of 2013, ~30 analyses



Bao-An Li and Xiao Han,
Phys. Lett. B727 (2013) 276

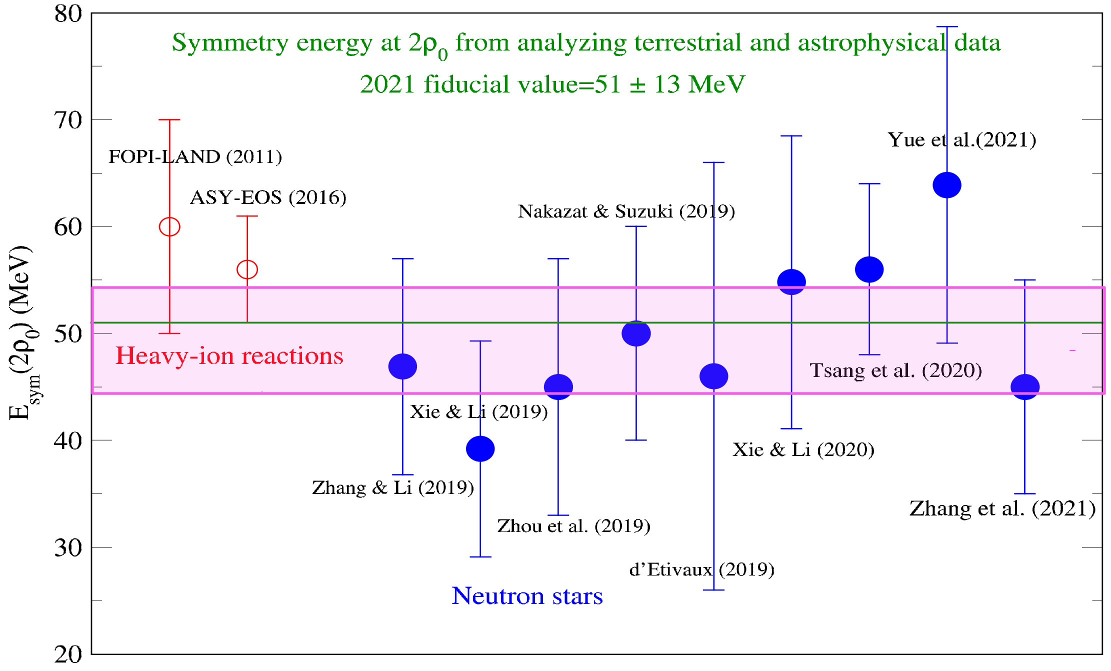
Slope L of nuclear symmetry energy as of 2023

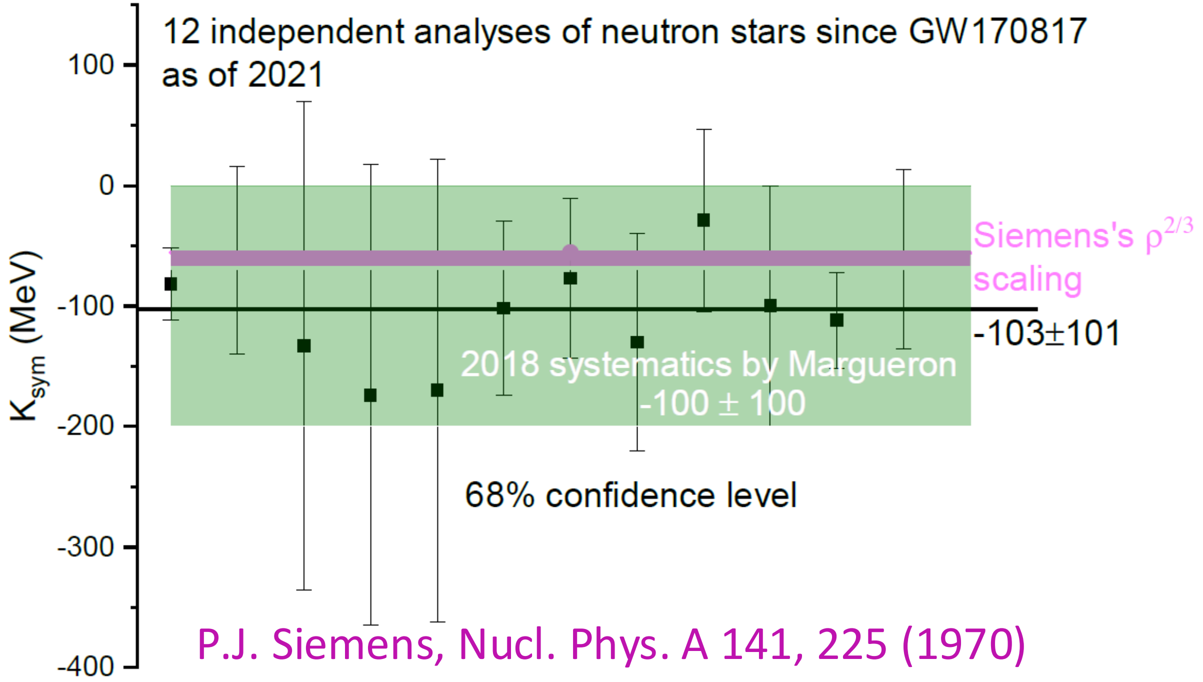
Even after 100 analyses, L remains broadly uncertain



Nai-Bo Zhang and Bao-An Li,
EPJA 59, 86 (2023)

Prior on $E_{sym}(2\rho_0)$ and K_{sym}: Despite many studies, constraints remain broad





Examples of recent theoretical predictions for $E_{sym}(2\rho_0)$:

(1) Chiral EFT, $E_{\mathrm{sym}}(2\rho_0) \approx 45 \pm 3\ \mathrm{MeV}$

C. Drischler, R. J. Furnstahl, J. A. Melendez, and D. R. Phillips, PRL**125**, 202702 (2020)

(2) Quantum Monte Carlo, $E_{\mathrm{sym}}(2\rho_0) \approx 46 \pm 4$

D. Lonardoni, I. Tews, S. Gandolfi, and J. Carlson, Phys. Rev. Research **2**, 022033(R) (2020)

(3) Relativistic BHF in full Dirac space: 51.6 MeV

Sibo Wang, Hui Tong, Qiang Zhao, Chencan Wang, Peter Ring, Jie Meng, PRC 106 (2022) 2, L021305

(4) Relativistic BHF: ~ 53 MeV

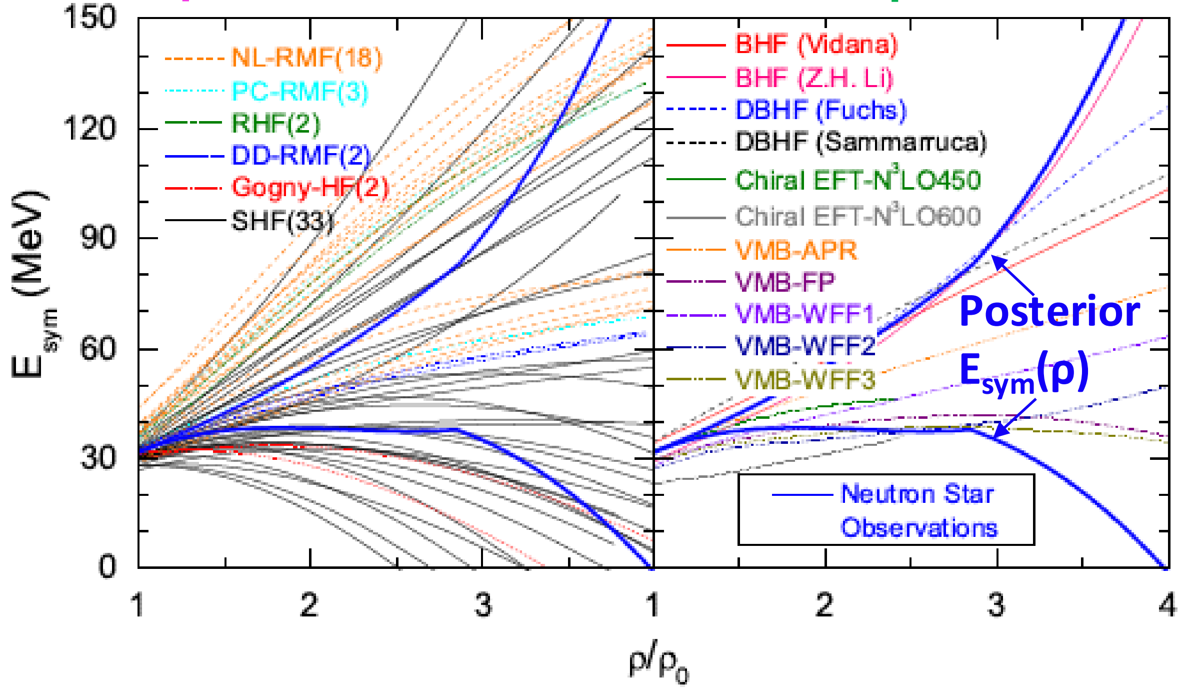
Chencan Wang, Jinniu Hu, Ying Zhang, Hong Shen, *Chin. Phys. C* 46 (2022) 6, 064108

Bao-An Li, Euro Phys. Journal Special Topic (2026)

Predictions diverge at high density

Phenomenological Models
60 examples

Microscopic & *ab initio* Theories
11 examples



L.W. Chen, Nucl. Phys. Rev. **34**, 20 (2017).
N.B. Zhang, B.A. Li, Eur. Phys. J. A **55**, 39 (2019).

Why is the high-density symmetry energy still so uncertain?

- Isospin-spin dependent Short-Range Correlations induced by tensor forces, many-body interactions
- New particles and possible phase transitions are important but poorly known

Why did the available neutron star data not help much at high densities?

- R is obtained from P®=0 at the surface, corresponding to an average density of about $2\rho_0$ where P_{sym} is larger than or compatible with P_{SNM}, depending mostly on K_{sym} and L

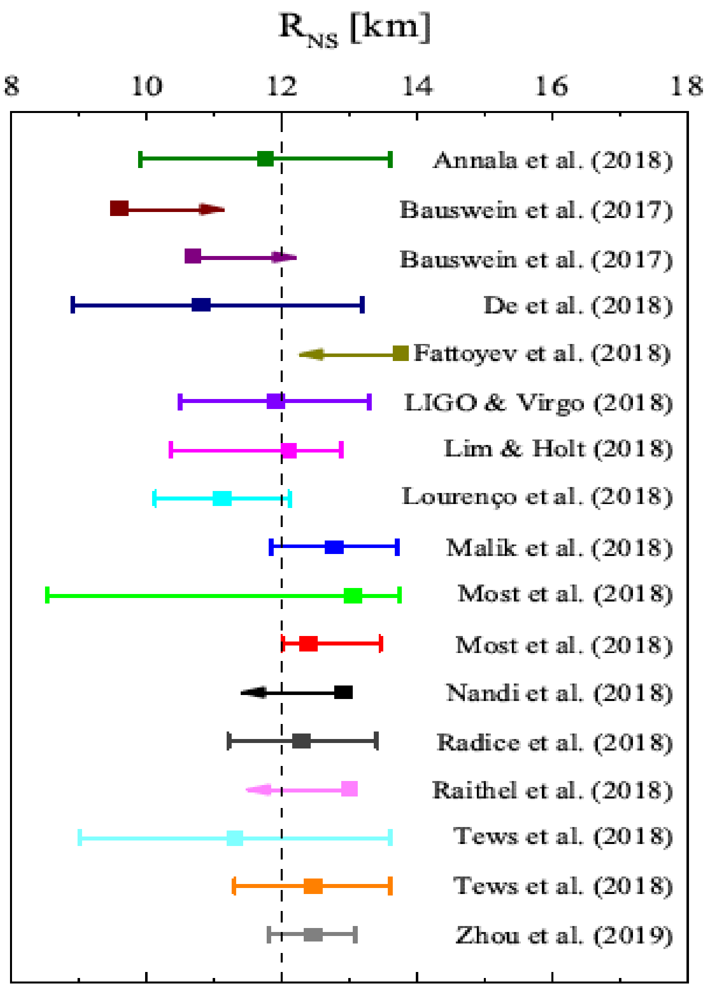
Current precision of neutron star radius measurements σ_R~1.0 km

Gravitational wave results

LIGO/VIRGO for GW170817: $R_{1.4} = 11.9 \pm 0.875$ km

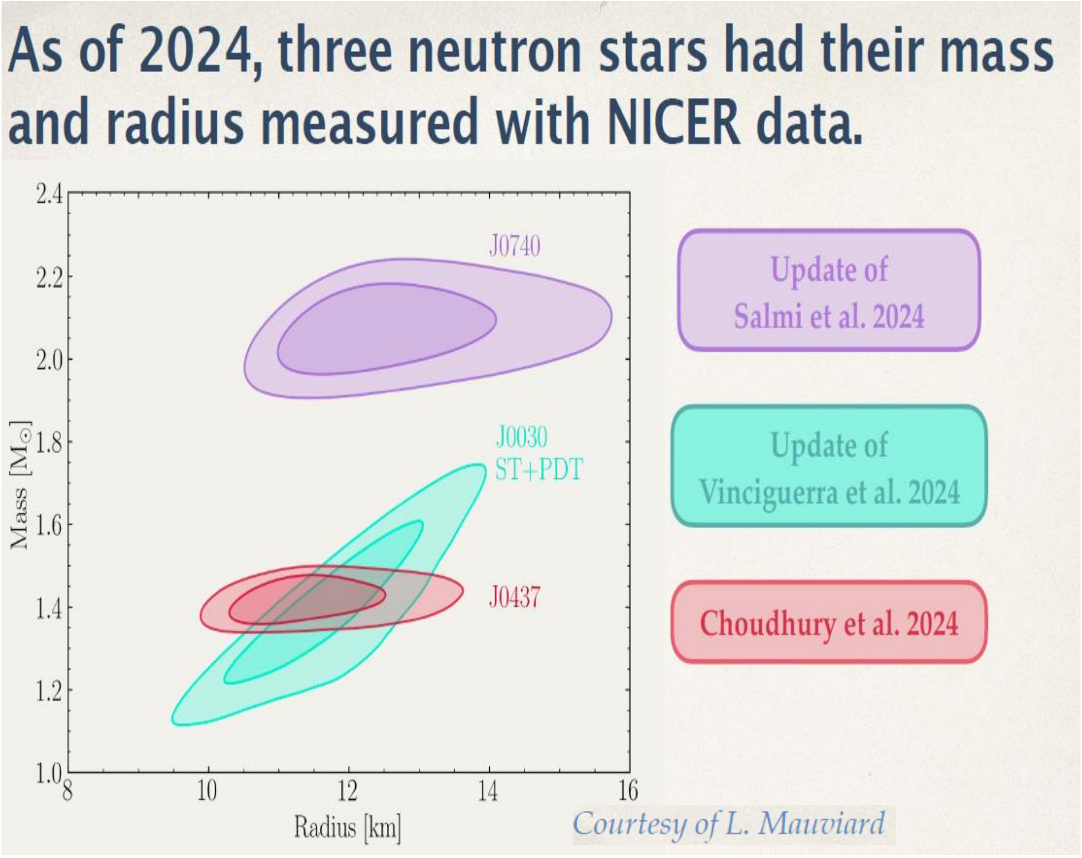
Various analyses during 2-years after GW170817

$R_{1.4} = 12.0 \pm 1.13$



B.A. Li et al., EPJA 55, 117 (2019)

X-ray results

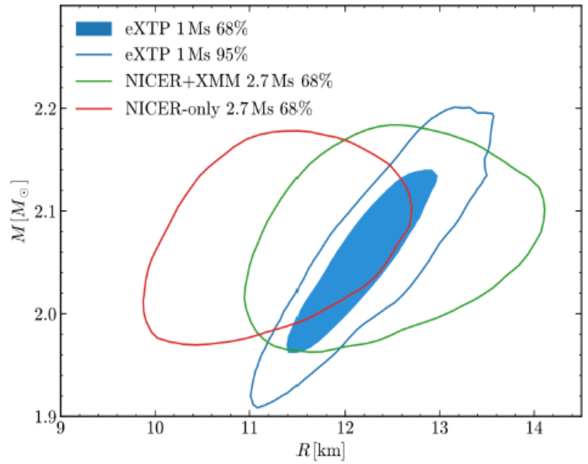


Future promise

eXTP: The enhanced X-ray Timing and Polarimetry mission

+/- 6% precision on radius

PSR J0740+6620



S. N. Zhang et al., arXiv:1812.04020v1
[Ang Li et al. 2506.08104](https://arxiv.org/abs/2506.08104) (2025 update)

NewATHENA

Advanced **T**elescope for **H**igh **En**ergy **A**strophysics

+/- 3% precision on radius

- **Radius 1-sigma uncertainties**
  - NICER 1600 ksec: ~10%
  - ATHENA 500 ksec: ~3% average (±0.3 km)

The NewAthena mission concept in the context of the next decade of X-ray Astronomy,
Nature Astronomy 9, 36 (2025)

Gravitational wave precision forecasts

Precision constraints on the neutron star equation of state with third-generation gravitational-wave observatories

Kris Walker[1,2,*] Rory Smith[3,†] Eric Thrane[2,3] and Daniel J. Reardon[4,5]

PHYSICAL REVIEW D **110**, 043013 (2024)

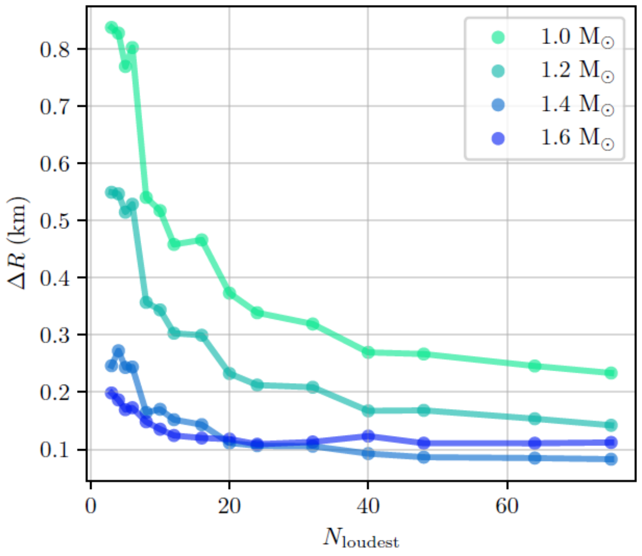
The Astrophysical Journal, 955:45 (8pp), 2023 September 20 https://doi.org/10.3847/1538-4357/acf12f

Prospects for a Precise Equation of State Measurement from Advanced LIGO and Cosmic Explorer

Daniel Finstad[1,2,3], Laurel V. White[4], and Duncan A. Brown[4]

Cosmic Explorer
+
Einstein Telescope

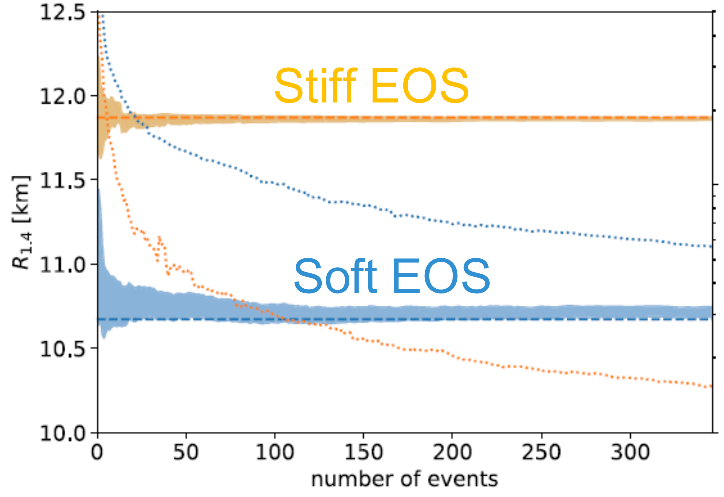
1-year joint operation



0.1 km

Number of loudest NS mergers

1-year Cosmic Explorer operation



0.6%

Relative error

0.1%

Bayesian inference of EOS from neutron star observables

Meta-EOS Model → Data → Posterior

Generating randomly within
Prior ranges of hadronic EOS

Parameters	Lower limit	Upper limit (MeV)
K_0	220	260
J_0	-800	400
$K_{\rm sym}$	-400	100
$J_{\rm sym}$	-200	800
L	30	90
$E_{\rm sym}(\rho_0)$	28.5	34.9

$$K_0,\ J_0\ ,L,\ K_{sym},\ J_{sym}\ ,\ E_{sym}(\rho_0)$$

Additional parameters for phase transition and quark matter

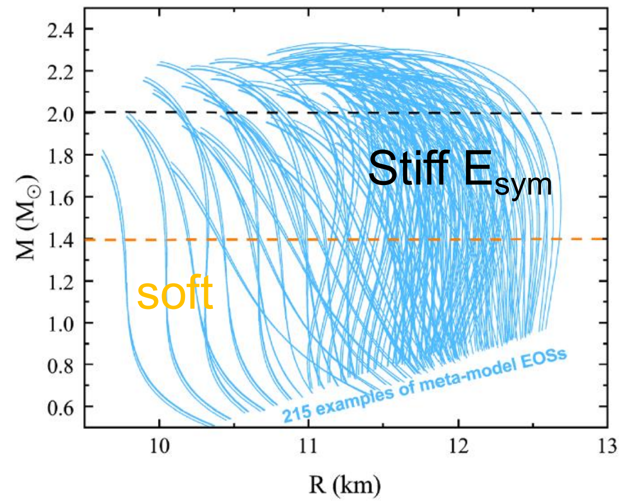
NS EOS

TOV equations

$$\frac{dP}{dr} = -\underbrace{\frac{G M \varepsilon}{r^2}}_{\text{Newtonian}} \overbrace{\left(1+\frac{P}{\varepsilon c^2}\right)}^{\text{matter correction}} \overbrace{\left(1+\frac{4\pi r^3 P}{Mc^2}\right)}^{\text{matter-geometry coupling}} \underbrace{\left(1-\frac{2GM}{rc^2}\right)^{-1}}_{\text{geometry correction}};\quad \overbrace{\frac{dM}{dr} = 4\pi r^2 \varepsilon / c^2}^{\text{same as Newtonian}}$$

matter+geometry corrections: ≫1

$M_{max} \geq 1.97 M_{sun}$



R-M relation

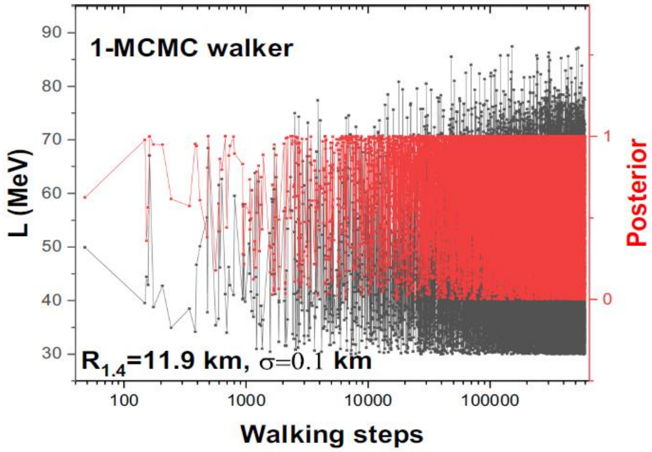
Using high-precision mock radius data here

$$\prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_{{\rm obs},j}} \exp\left[-\frac{(R_{{\rm th},j} - R_{{\rm obs},j})^2}{2\sigma^2_{{\rm obs},j}}\right]$$

Likelihood function (prediction vs data)

Markov Chain Monte Carlo (MCMC)

P(D|M)

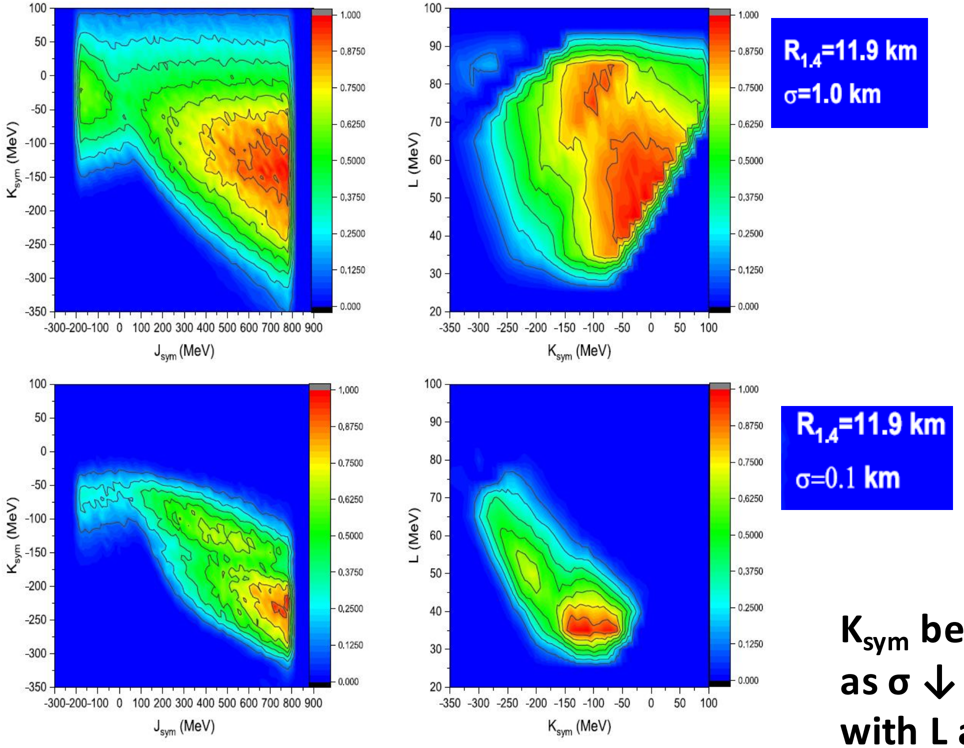


The proposed high-precision measurements of neutron star radii are SUPER-expensive in Time & Efforts

What do we expect high-precision R to constrain?

Bao-An Li, Xavier Grundler, Wen-Jie Xie, Nai-Bo Zhang, PRD 110, 103040 (2024)

Using mock radius data: $R_{1.4}$=11.9 $\pm\sigma$



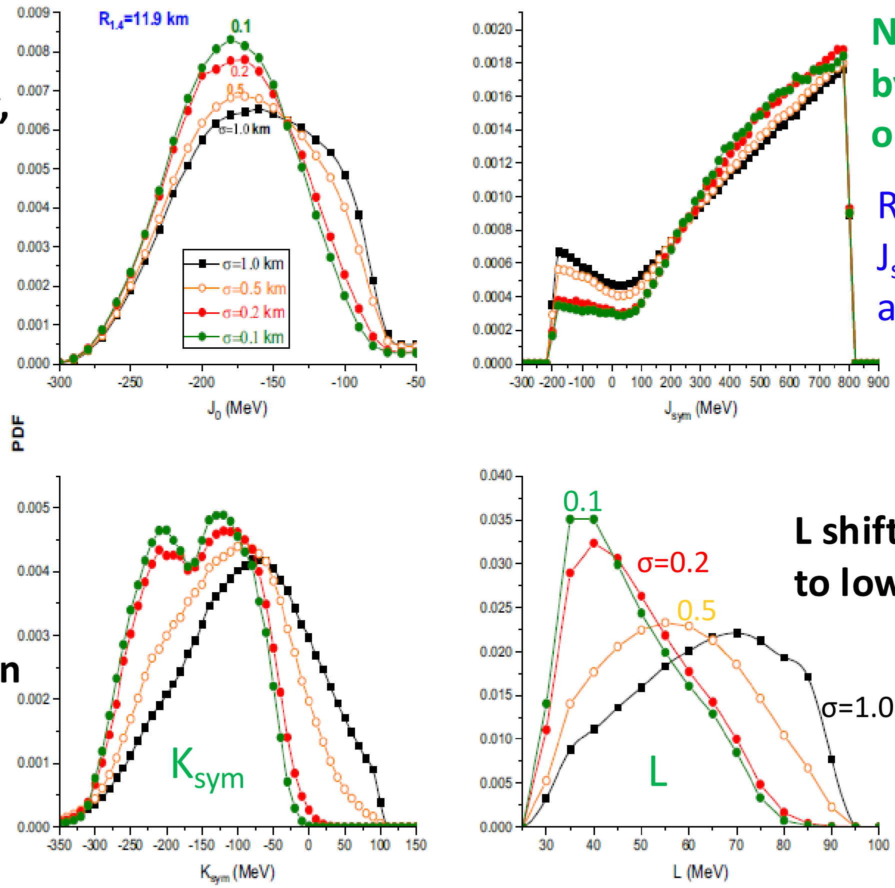
Constrained by M_{TOV} & casualty, not much by R

K_{sym} becomes bimodal as σ ↓ duo its correlation with L and J_{sym}

Strong EOS-parameter correlations appear at higher precisions

Skewness J_0 of SNM

Skewness J_{sym} of E_{sym}



NOT constrained by R data, regardless of its precision

R probes EOS ~2ρ_0, J_{sym} characterizes E_{sym} at ρ>3ρ_0

L shifts systematically to lower values as σ ↓

Nonlinear mapping + uncertainty = systematic inference shift (Jensen bias)

Jensen decision bias (economics, statistics), noise-induced drift in nonlinear dynamics (biology, chemistry and physics)

Mathematical proof: Johan Jensen, *Acta Math.* 30, 175 (1906).

Observable x (e.g., $R_{1.4}$) fluctuating around its mean x_0 like a Gaussian: $x = x_0 + \delta x$ with variance σ^2.

Mapping function f(x) (pressure or any EOS parameter): $f(x) = f(x_0) + f'(x_0)\delta x + \frac{1}{2}f''(x_0)(\delta x)^2 + \cdots$.

The mean value of f($\langle f(x)\rangle \approx f(x_0) + \frac{1}{2}f''(x_0)\sigma^2$ Jensen bias:

- Convex functions ($f'' > 0$) produce upward shifts.
- Concave functions ($f'' < 0$) produce downward shifts.

Uncertainty in x doesn't just make answers about f(x) fuzzier—it actually **shifts** the most likely answer away from the center

Example: Utility function of wealth = Happiness from wealth = psychological value of wealth: U(M)=Log(Money)

First proposed by Daniel Bernoulli in 1738: *"Specimen Theoriae Novae de Mensura Sortis"* (1738) translated as *"Exposition of a New Theory on the Measurement of Risk"*, Econometrica **22**, 23 (1954)

Adopted in behavioral economics, decision theory, neuroscience: U(M)=log(Money) or $\sqrt{Money}$ (concave), not M itself, because most people make economic decisions based on the relative uncertainty of M, δLog(M)=δM/M, not δM itself

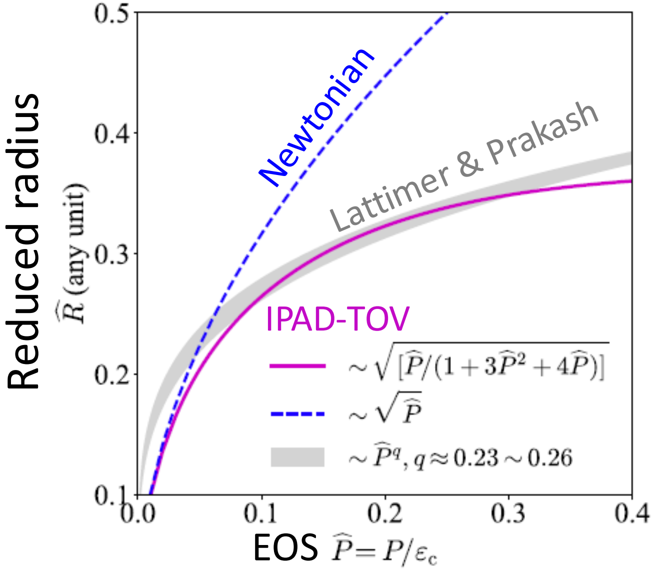
$U(M) = \log M,$ ➡ $U''(M) = -\frac{1}{M^2} < 0,$ ➡ $\langle \log M\rangle \approx \log M_0 - \frac{\sigma^2}{2M_0^2}$ ➡ **Risk Aversion**

Uncertainty in M lowers the expected utility even when the average wealth M_0 remains unchanged, as $M=M_0 \pm \sigma$

Mas-Colell, Whinston and Green, Microeconomic Theory (Oxford University Press, 1995)

TOV equations are highly nonlinear → Pressure(Radius) mapping function is highly convex at all densities

Forward mapping

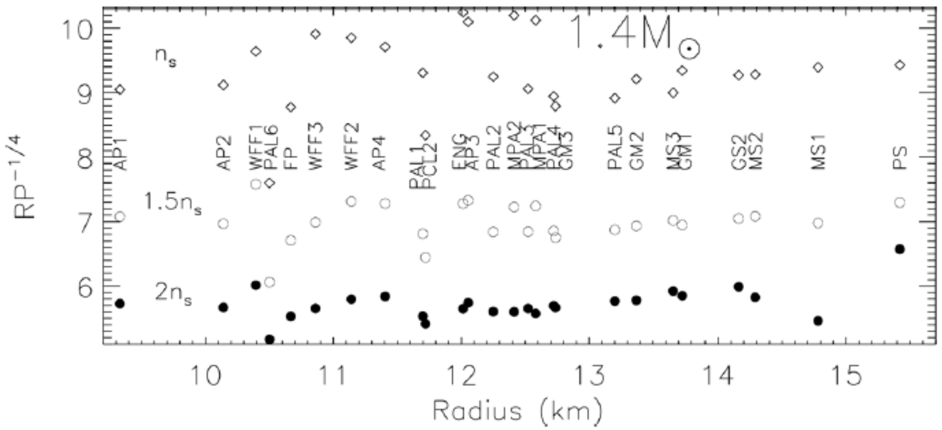


$$R\sim\nu_c\equiv\frac{\hat{P}_c^{1/2}}{\sqrt{\varepsilon_c}}\left(\frac{1}{1+3\hat{P}_c^2+4\hat{P}_c}\right)^{1/2}$$

Where $\hat{P}\equiv P/\varepsilon_c$ and ε_c is the central energy density.

Inverse mapping

Pressure $P(R_{1.4},\rho)\sim[R_{1.4}/c(\rho)]^4$



Lattimer & Prakash, Phys. Rep., 442, 109 (2007)

At $n_s=\rho_0$, $P=P_{\rm sym}(\rho_0)\propto\frac{\rho_0 L}{3}$.

The P(ρ) inferred from high-precision R-data can fix L

At $n_s=2\rho_0$ $\quad a>b\gg c.$

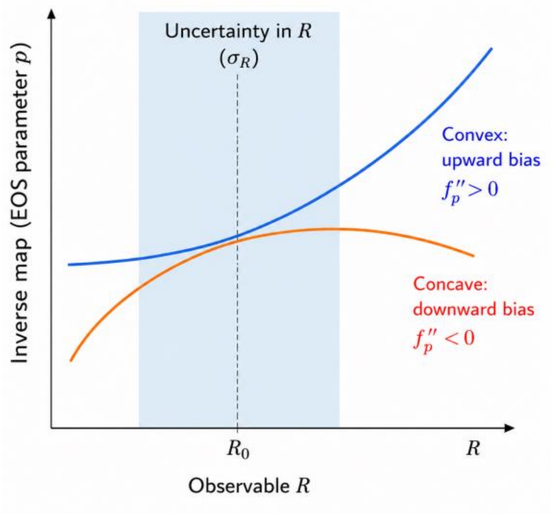
$$P(2\rho_0)=P_{\rm SNM}+P_{\rm sym}\sim aL+bK_{\rm sym}+cJ_{\rm sym},$$

(1) High-precision R-data can fix the combination of L, K_{sym} and J_{sym},

(2) after L is sufficiently fixed, K_{sym} can be constrained, and it may show a bimodal structure because of its correlation with L and J_{sym}

(3) J_{sym} is very weakly constrained by R data

Convex mapping shifts EOS parameters upward with an increasing radius uncertainty squarely



IPAD-TOV: Intrinsic and Perturbative Analysis of Dimensionless TOV

Analogy:
iPad: Get online by touching without using a hard keyboard
IPAD-TOV: Get radius-pressure scaling without using any input EOS

Bao-Jun Cai, Bao-An Li and Zhen Zhang,

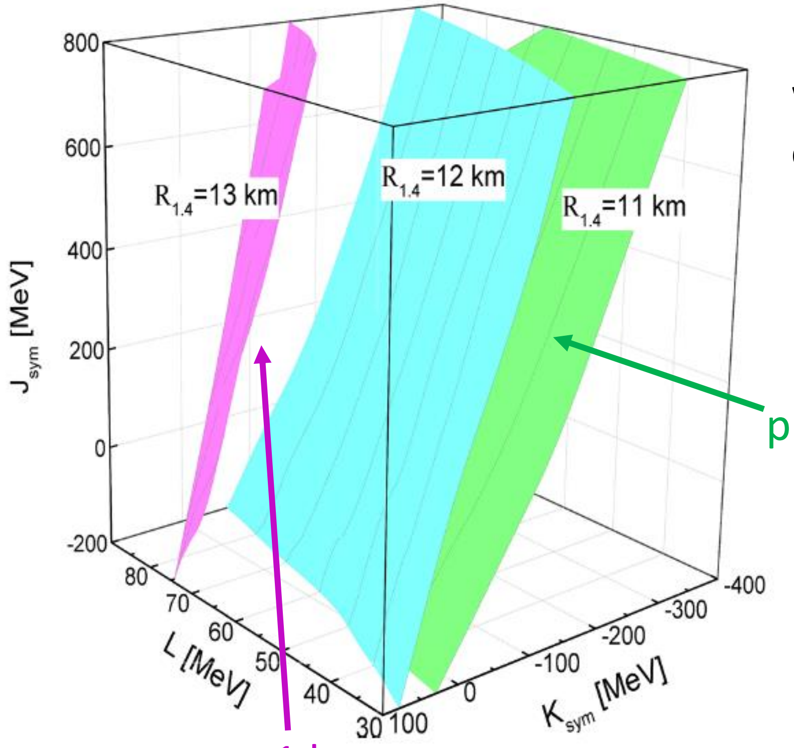
ApJ 952,147(2023); PRD 108,103041(2023)

Consistent geometric and analytical interpretation of R-precision-dependent Bayesian inference of EOS parameters

Highly nonlinear TOV equations → Unequal mapping of EOS population

Equal radius intervals $\pm\sigma_R$ on the two sides of R_0 correspond to highly unequal volumes in EOS parameter space

Constant-radius surfaces



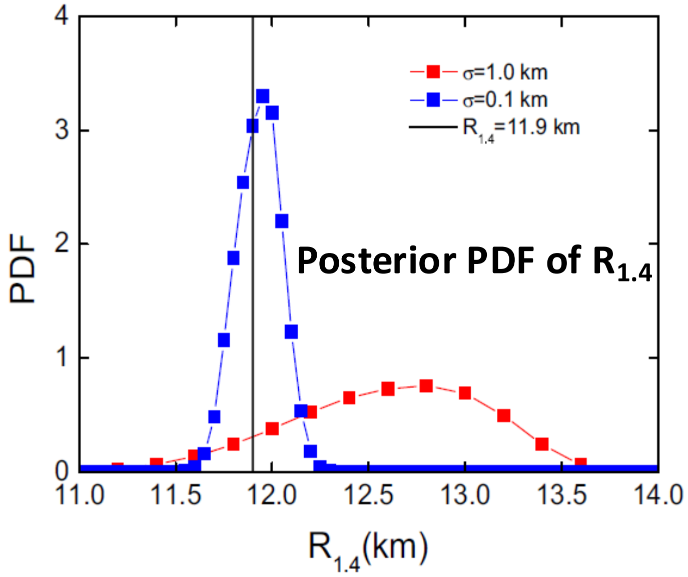
With large σ, Bayesian integral samples disproportionately more stiff E_{sym},
→ large-radius configurations
→ posterior $R_{1.4}$ peaks upward

-1 km
physically restricted regions

+1 km
"more open" EOS space toward larger radii

N.B. Zhang and B.A. Li, JPG 46, 014002 (2019)

Larger observational uncertainty biases the posterior radius upward



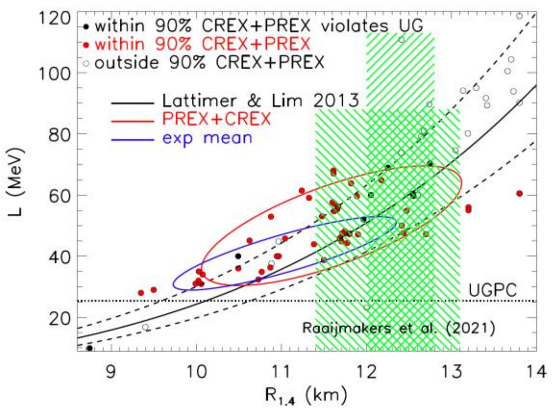
B.A. Li et al., PRD 110, 103040 (2024)

$$\langle P(R_{1.4})\rangle = P(R_0) + \frac{1}{2}P''(R_0)\sigma_R^2 + \cdots .$$

$$P(2\rho_0) \sim R_{1.4}^4. \longrightarrow P''(R_0) = 12R_0^2 > 0$$

$$\langle P(R_{1.4})\rangle > P(R_0 = 11.9\text{ km}).$$ unless σ_R=0

Understanding the posterior PDFs of EOS parameters with varying precision σ



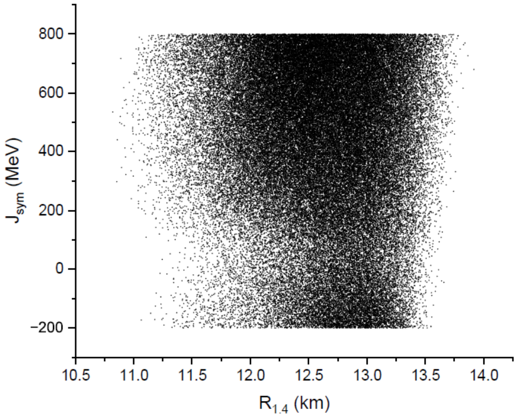
$$R_{1.4} \simeq (9.51 \pm 0.49)(P_{NSM}/\mathrm{MeV\ fm^{-3}})^{1/4}\ \mathrm{km}.$$

$$P_{NSM}(n_s) = \left(n^2 \frac{\partial E_{NSM}}{\partial n}\right)_{n_s} \simeq \frac{L n_s}{3}\left[1 - \left(\frac{4J}{\hbar c}\right)^3 \frac{4 - 3J/L}{3\pi^2 n_s}\right].$$

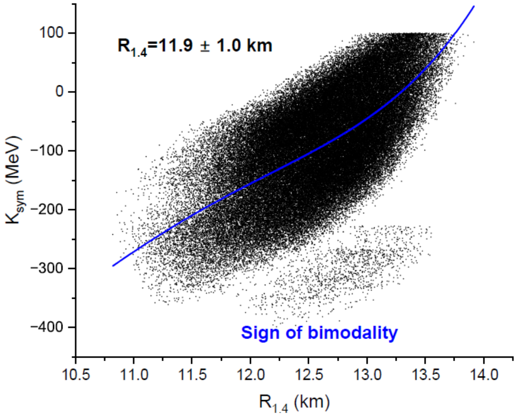
PNM   ß-equilibrium correction ~ 0.1

James Lattimer, Particle 6, 30 (2023)

Convex L, k_{sym} vs $R_{1.4}$ mapping



9526 samples from 1/24 MCMC-walkers



Understanding qualitatively effects of R precision on PDF(L): Assume L ~ R^4 (valid as long as it is convex)

(1) Mean of L shifts upward with increasing σ^2,

$$\mathrm{PDF}(R) \propto \exp\left[-\frac{(R-R_0)^2}{2\sigma^2}\right], \quad L = R^4.$$

$$R = R_0 + \delta,$$

$$\langle\delta\rangle = 0, \quad \langle\delta^2\rangle = \sigma^2.$$

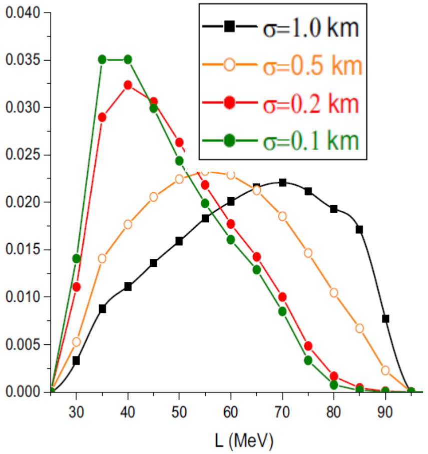
$$\langle R^4\rangle = R_0^4 + 6R_0^2\sigma^2 + 3\sigma^4.$$

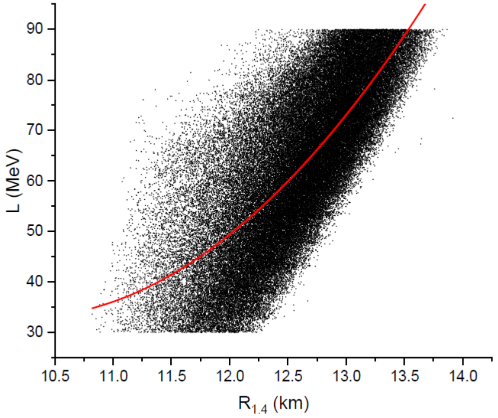
(2) Inferred L probability distribution is skewed

$$\mathrm{PDF}(L) = \mathrm{PDF}(R)\left|\frac{dR}{dL}\right|.$$

$$R = L^{1/4}, \quad \frac{dR}{dL} = \frac{1}{4}L^{-3/4},$$

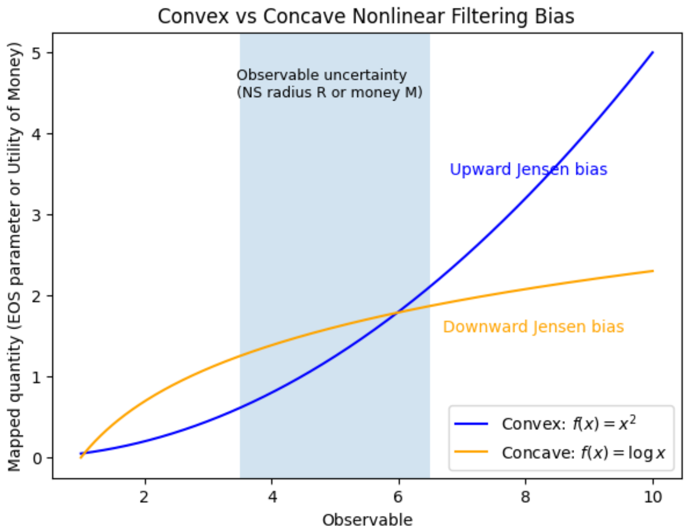
$$\mathrm{PDF}(L) \propto \exp\left[-\frac{(L^{1/4}-R_0)^2}{2\sigma^2}\right] L^{-3/4}$$





Bao-An Li, et al., PRD 110, 103040 (2024)

Nonlinear Filtering & Precision Dependence



Economics

Wealth M

Utility U(M)=Log(M)

Concavity from human psychology → downward bias

Risk aversion

Neutron star physics

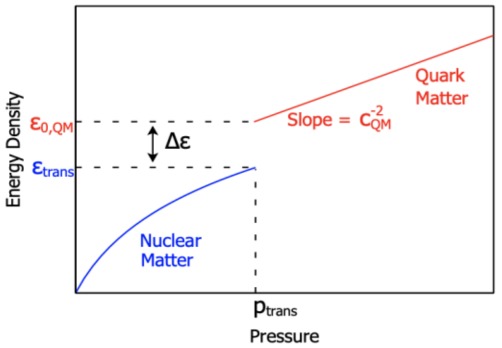
Radius R

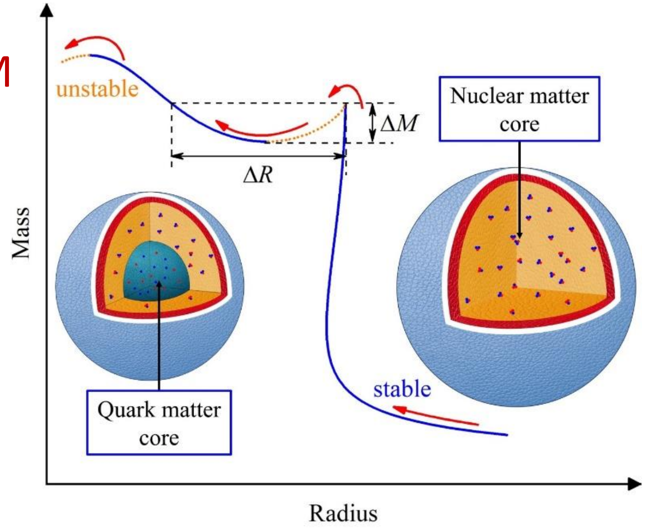
EOS $P(R) \sim [R/c(\rho)]^4$

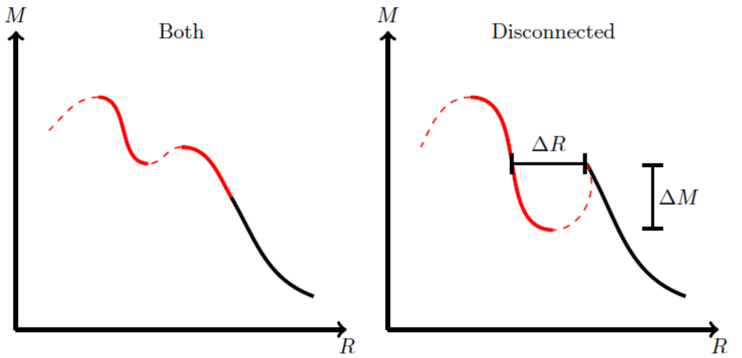
Convexity from neutron star nature (TOV) → upward bias

Stiff-EOS preference under broad uncertainty

Hybrid EOS with a 1st order hadron-quark PT with a constant speed of sound model for QM





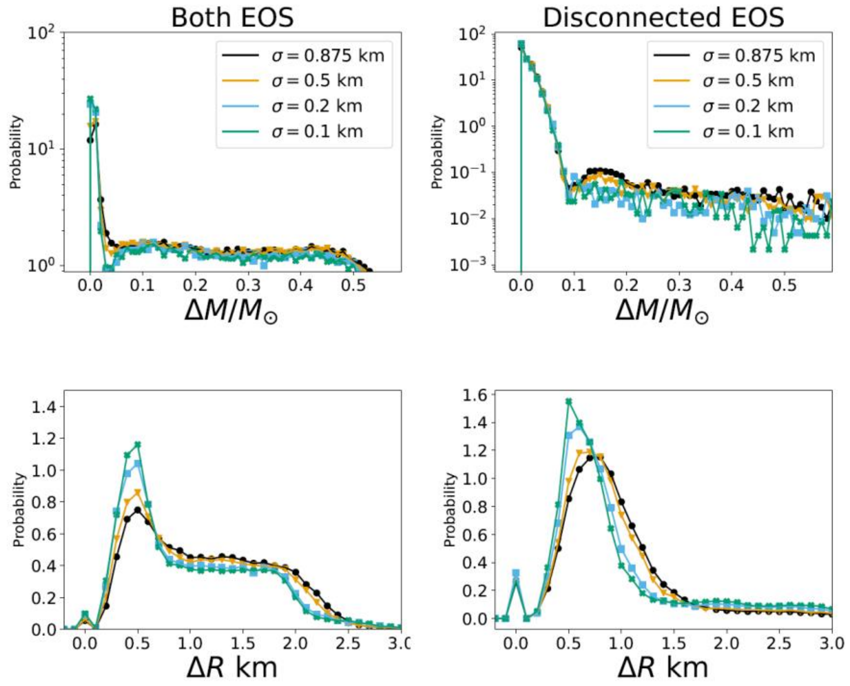


M.G. Alford, S. Han, M. Prakash, PRD 88, 083013 (2013)

(1) No confirmed observation yet, most analyses require low transition density

(2) Possible without a phase transition, R. Essick, APJ L 973, L50 (2024)

Formation probability



Bayesian quantification of the observability of twin stars from future high-precision R data

Xavier Grundler and Bao-An Li, PRD 112, 103012 (2025)

Using a 9-parameter meta-model EOS

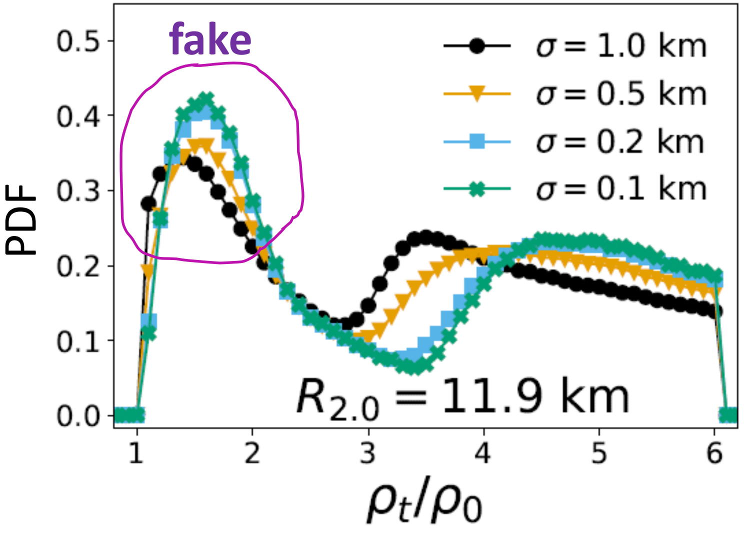
Prior range of ρ_t: (1-6)ρ_0

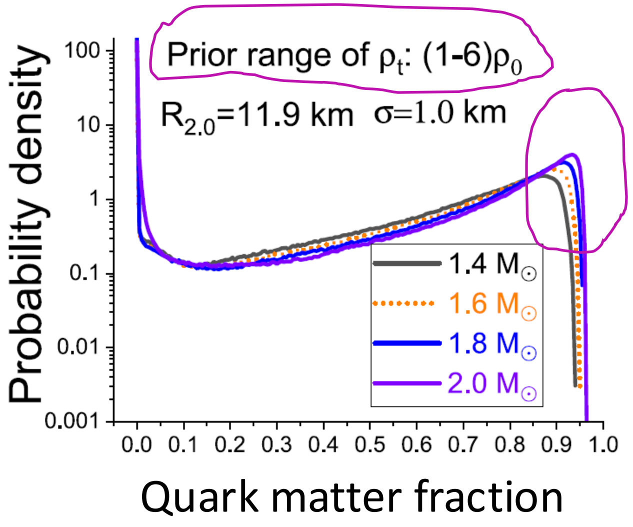
Using mock radius data: $R_{1.4}$=11.9 $\pm\sigma$

Can the high-precision R tell us anything new about high-density EOS?

Bao-An Li, Xavier Grundler, W.J. Xie, N.B. Zhang, APJ 998, 262 (2026)

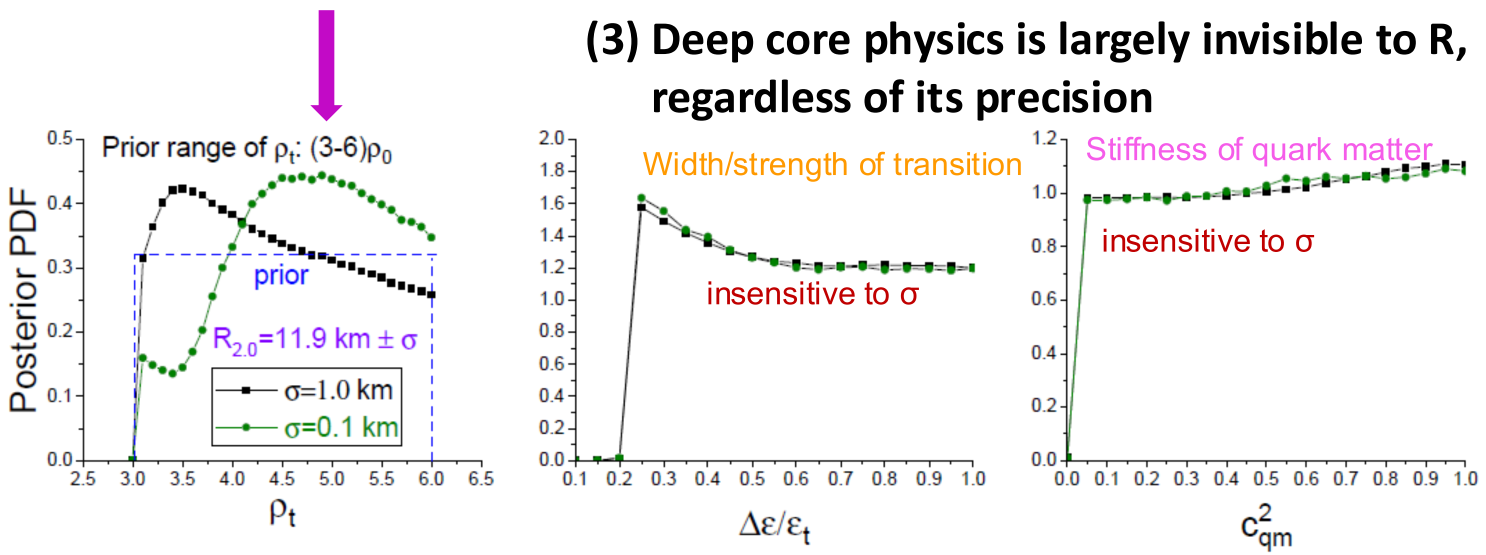
(1) Pure statistics can NOT replace physics insight; existing NS data are not sufficient to constrain all parameters





Consistent with indications of Beam Energy Scan Experiments at RHIC

(3) Deep core physics is largely invisible to R, regardless of its precision



(2) Effects on hadron-quark transition density

Low ρ_t produces strong EOS softening, large σ tolerates these softer solutions, and shifts the hadronic pressure to higher values (Jensen's bias)

High-precision R data prefer delayed quark deconfinement

Do not probe the quark EOS itself

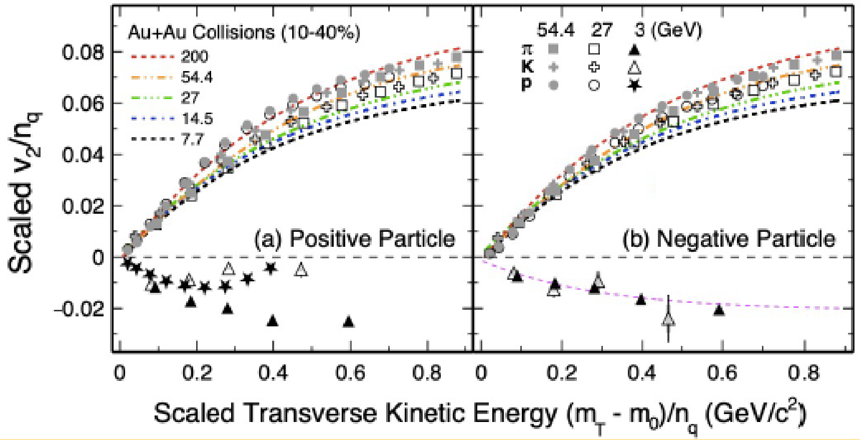
BES/STAR indications on hadron-quark transition density in hot dense matter

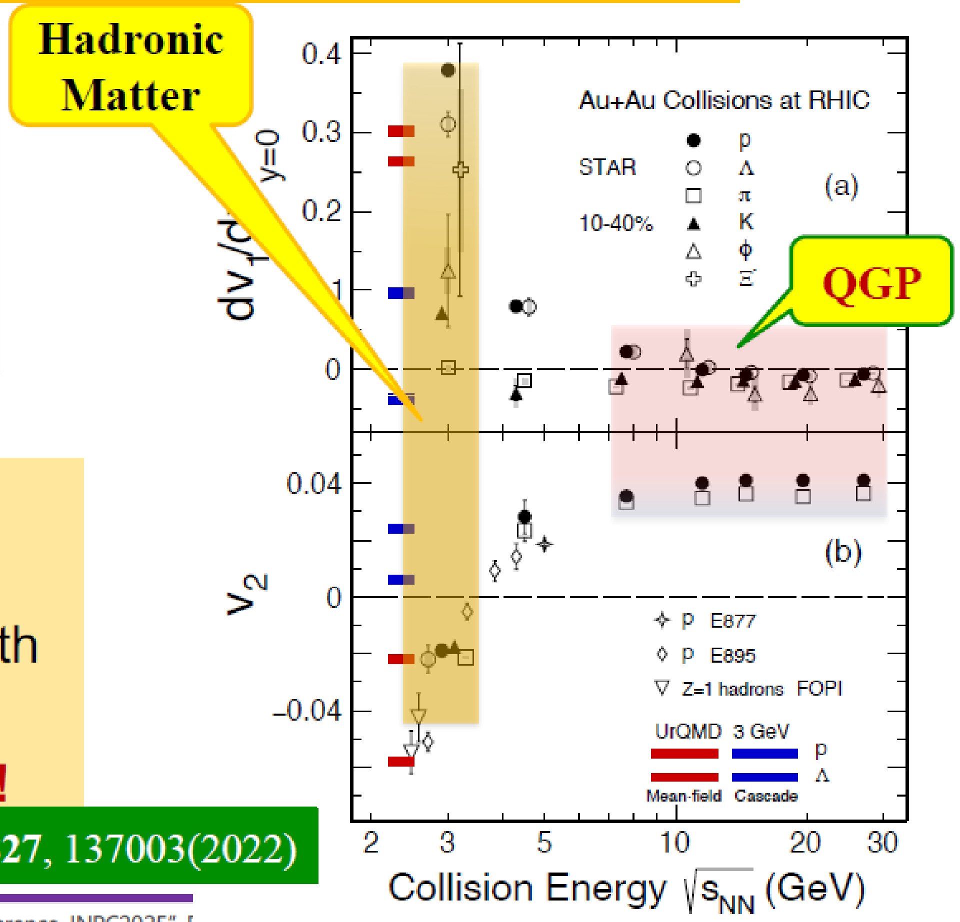
Onset of partonic collectivity in Au+Au at 4.5 GeV

STAR: Phys. Rev. Lett. 135, 072301 (2025)

Partonic Collectivity or Not

INPC 2025

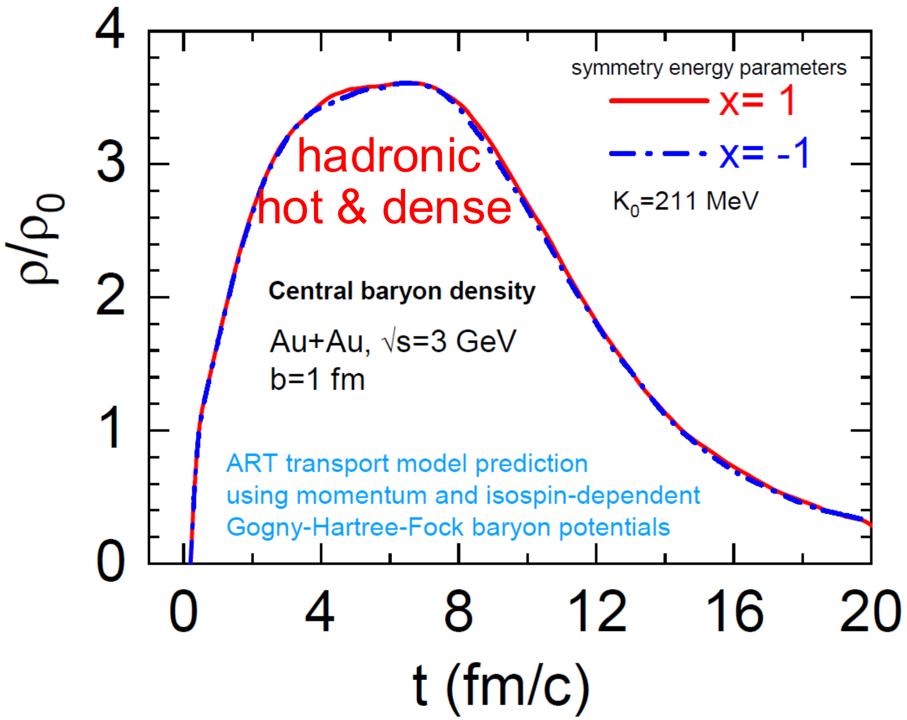




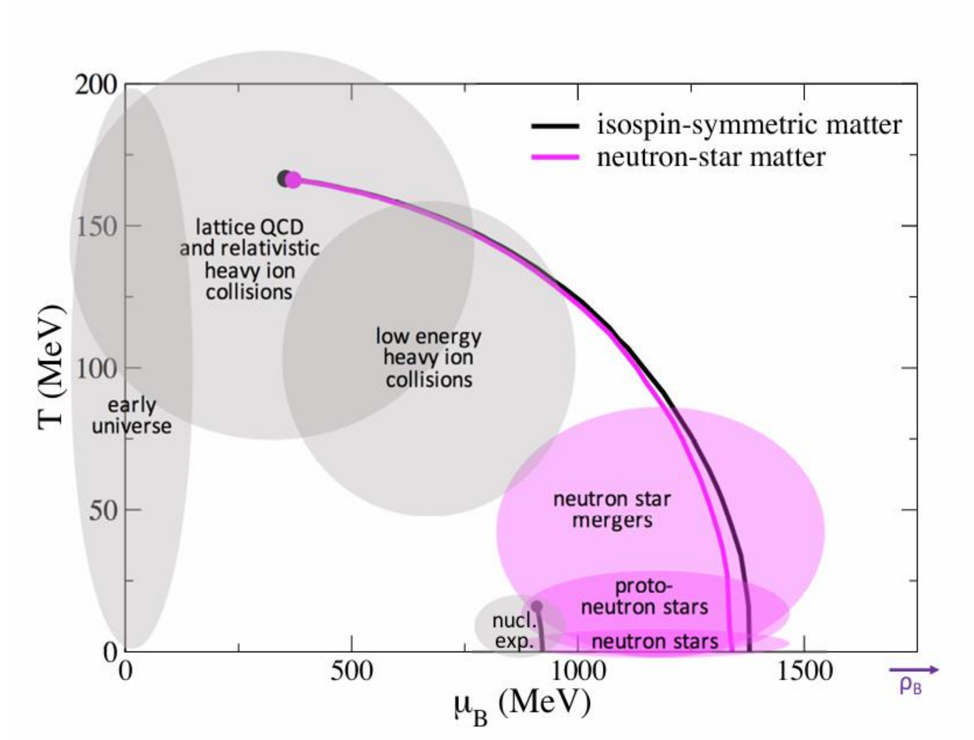
- At **3 GeV**, NCQ scaling is absent;
- Transport model calculations, with baryonic mean field, reproduce both v_1 and v_2 results;
- **Hadronic interactions dominant!**

STAR: PLB**827**, 137003(2022)

BES/STAR Implications for hadron-quark transition density in cold neutron stars



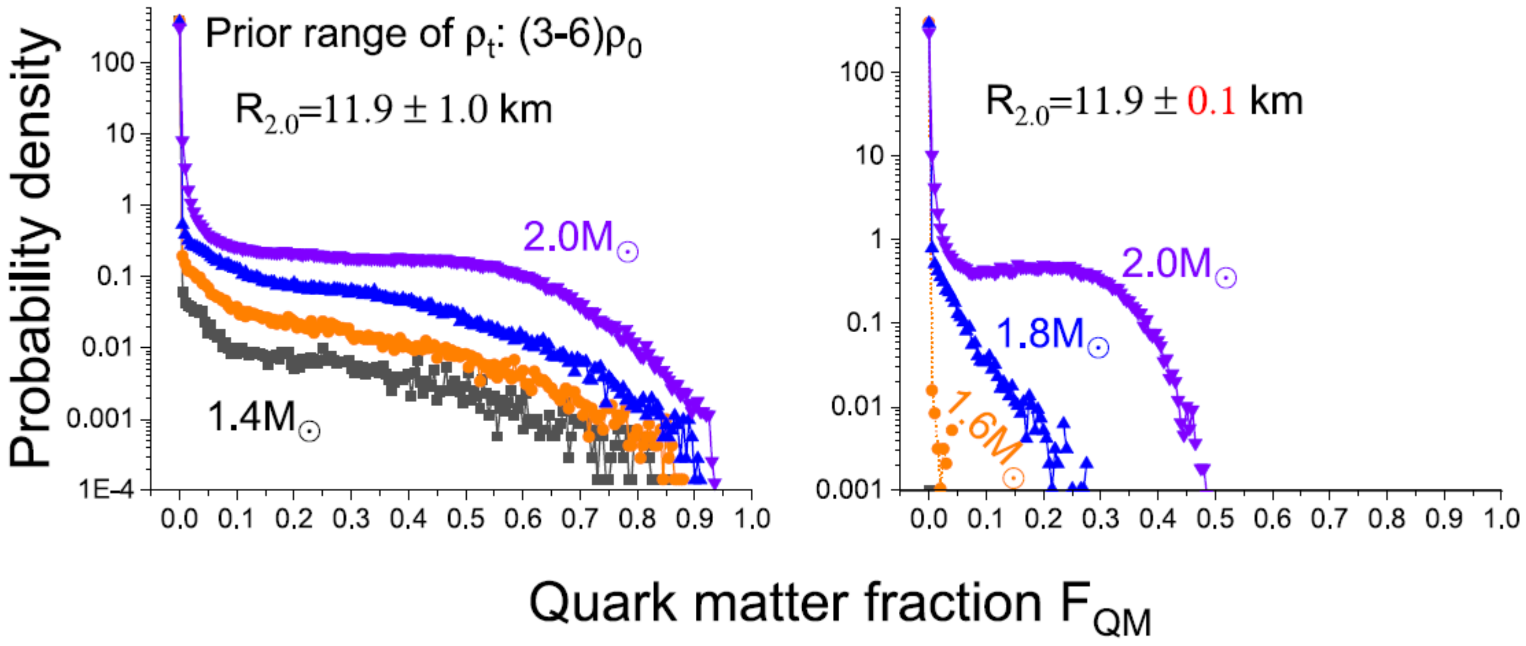
G.C. Yong, B.A. Li, Z.G. Xiao and Z.W. Lin, PRC **106**, 024902 (2022)



Veronica Dexheimer et al., *IJMPE* 27 (2018) 11, 1830008

The 1st order hadron-quark phase transition is unlikely below about $3.6\rho_0$ in neutron stars

➡ **Modify the prior lower-boundary of the hadron-quark transition density**



Prior range of ρ_t: (3-6)ρ_0
R_2.0=11.9 ± 1.0 km
2.0M⊙
1.4M⊙
Probability density
R_2.0=11.9 ± 0.1 km
2.0M⊙
1.8M⊙
1.6M⊙
Quark matter fraction F_QM

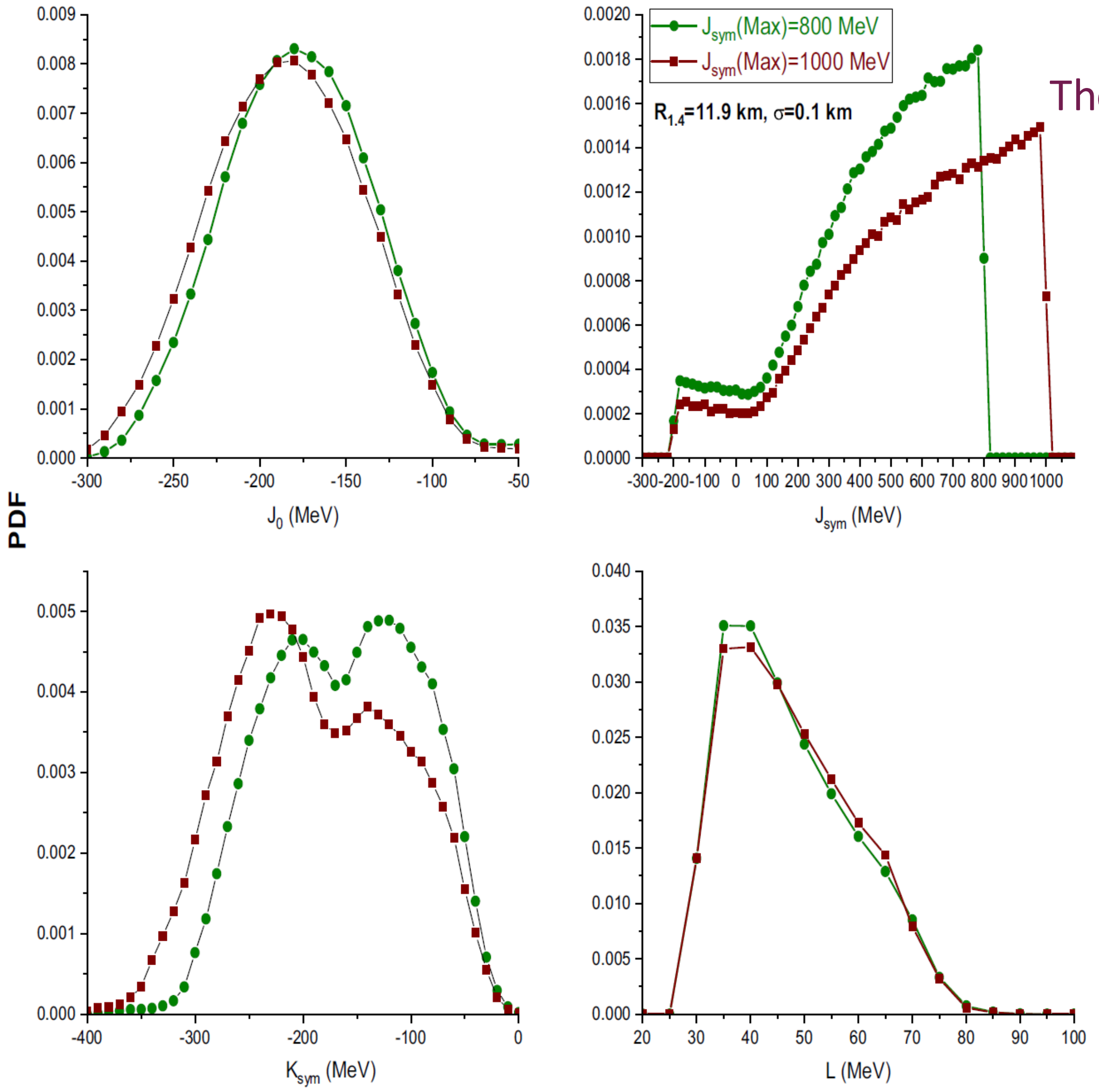
Take-Home Messages

- High-precision neutron-star radius measurements mainly constrain the EOS around ~$2\rho_0$, especially the symmetry-energy slope L and curvature K_{sym}, while deep-core quark-matter properties remain weakly constrained.
- Because the TOV equations are highly nonlinear, observational uncertainties produce systematic Bayesian inference shifts (Jensen-like bias), affecting both the locations and shapes of posterior EOS parameter distributions.
- Future precise radius measurements suppress early EOS softening and favor delayed hadron–quark deconfinement, constraining when phase transitions occur more strongly than the detailed quark EOS itself.

Future high-precision R measurements will strongly constrain the intermediate-density EOS and the onset of phase transitions, but deep-core physics still requires multimessenger astrophysical observations and nuclear laboratory experiments.

Back-up slides

Effects of increasing the prior upper limit of J_{sym}



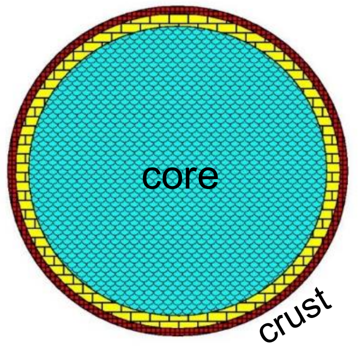
The NS data used is not constraining J_{sym}

Symmetry energy controls composition and affects pressure in neutron stars

(1) The proton (electron) fraction $x(Y_e)$ is determined by the $E_{sym}(\rho)$ through charge neutrality and beta-equilibrium conditions:

$$x = 0.048[E_{sym}(\rho)/E_{sym}(\rho_0)]^3(\rho/\rho_0)(1-2x)^3$$

Critical for the cooling mechanism of protoneutron stars and associated neutrino emissions, appearance of hyperons, kaon condensation, baryon resonances…..



(2) The pressure in the npe matter at beta equilibrium:

$$P(\rho,\delta) = \rho^2[\frac{dE_0(\rho)}{d\rho} + \frac{dE_{\rm sym}(\rho)}{d\rho}\delta^2] + \frac{1}{2}\delta(1-\delta)\rho E_{\rm sym}(\rho)$$

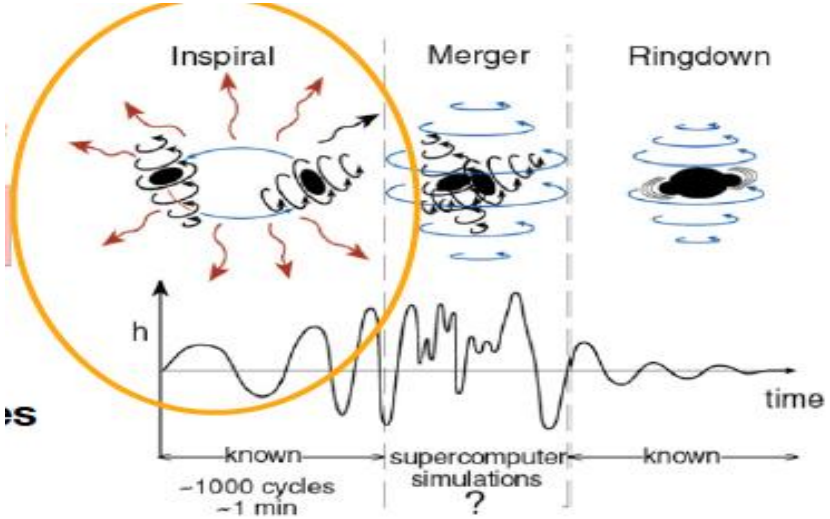
(3) The crust-core transition density and pressure is determined by setting the incompressibility of neutron star matter =0 (speed of sound becomes imaginary):

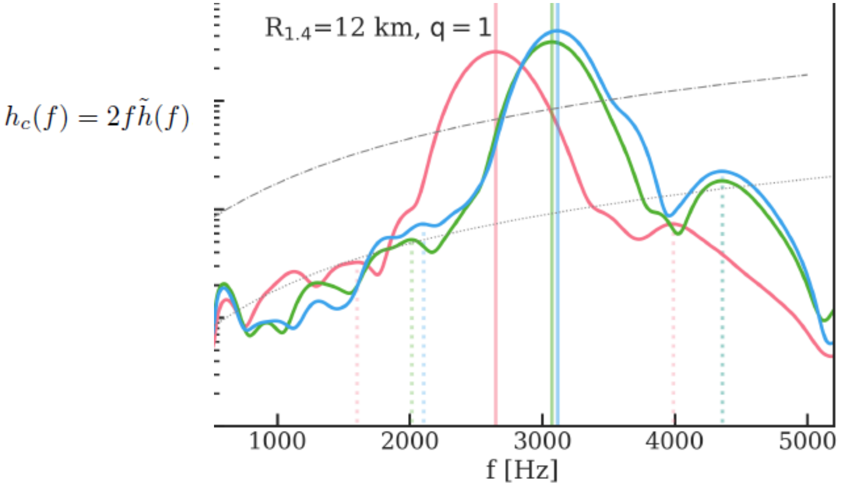
$$K_\mu = \rho^2\frac{d^2E_0}{d\rho^2} + 2\rho\frac{dE_0}{d\rho} + \delta^2\left[\rho^2\frac{d^2E_{sym}}{d\rho^2} + 2\rho\frac{dE_{sym}}{d\rho} - 2E_{sym}^{-1}(\rho\frac{dE_{sym}}{d\rho})^2\right] = 0$$

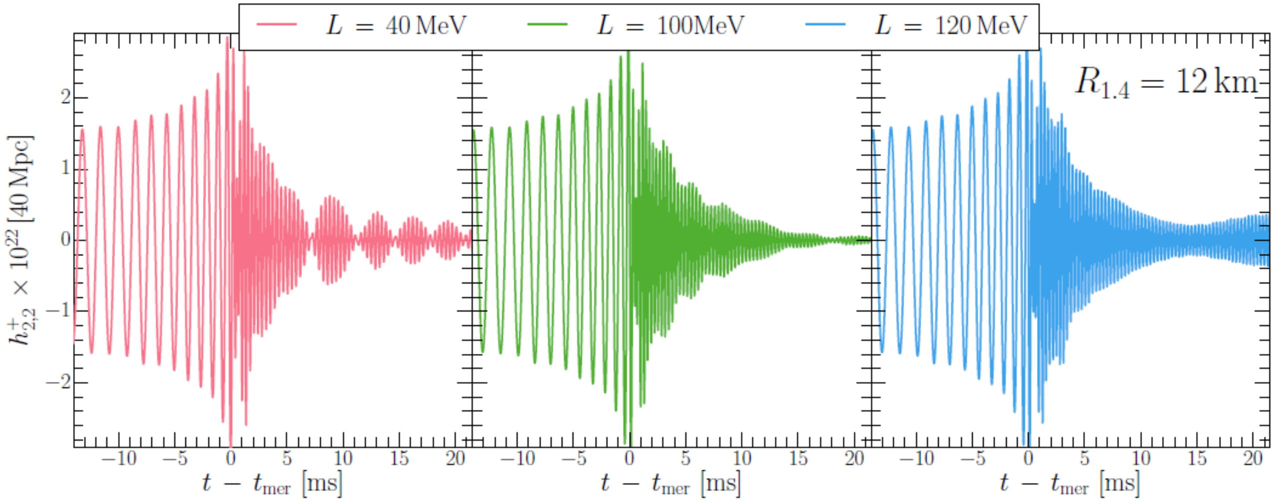
Lattimer & Prakash, Phys. Rep., 442, 109 (2007)

Impact of nuclear symmetry energy on the post-merger phase of a binary neutron star coalescence

Elias R. Most and Carolyn A. Raithel, PRD **104**, 124012 (2021)







The peak positon depends on L

Need to measure high-frequency GWs

nature communications

Article

https://doi.org/10.1038/s41467-023-43932-6

An updated nuclear-physics and multi-messenger astrophysics framework for binary neutron star mergers

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Check for updates

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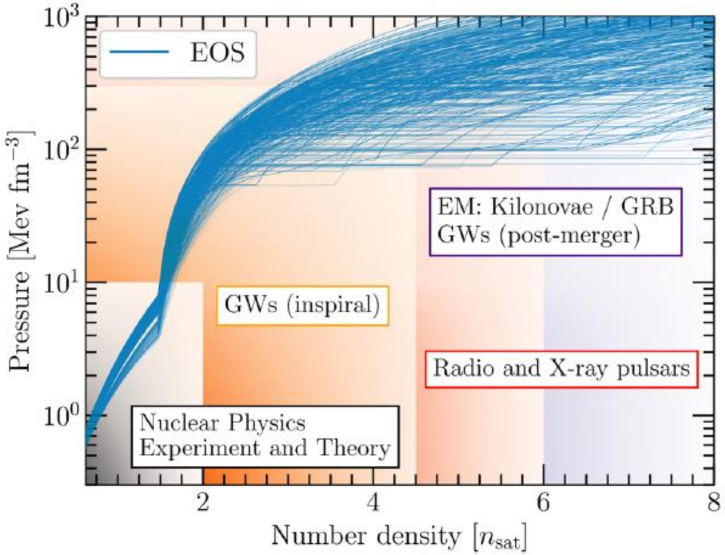


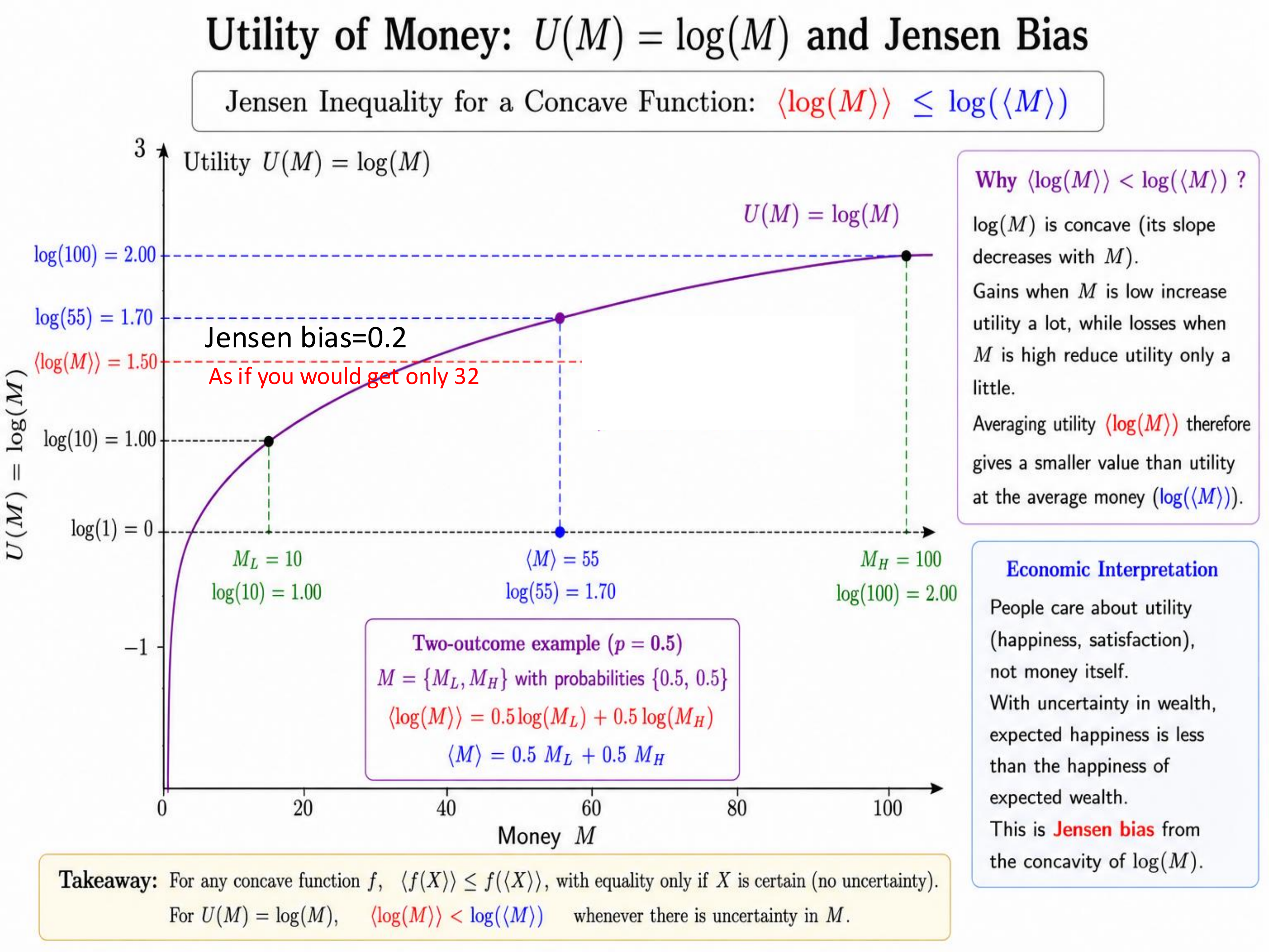
EOS=QMC+multi-segments of constant speed of sound at high densities

M. Bulla's transport code POSSIS for Kilonovae, MNRAS 489, 5037 (2019).

as a first attempt of analyzing the gravitational-wave signal, the kilonova, and the gamma-ray burst afterglow simultaneously. Incorporating all available information, we estimate the radius of a $1.4M_\odot$ neutron star to be $R = 11.98^{+0.35}_{-0.40}$ km.

LIGO/VIRGO; $R_{1.4}$=11.9±0.87 at 68% CFL

ChatGPT:


Utility of Money: U(M) = log(M) and Jensen Bias
Jensen Inequality for a Concave Function: ⟨log(M)⟩ ≤ log(⟨M⟩)
3
Utility U(M) = log(M)
U(M) = log(M)
log(100) = 2.00
log(55) = 1.70
Jensen bias=0.2
⟨log(M)⟩ = 1.50
As if you would get only 32
log(10) = 1.00
log(1) = 0
−1
U(M) = log(M)
M_L = 10
log(10) = 1.00
⟨M⟩ = 55
log(55) = 1.70
M_H = 100
log(100) = 2.00
Two-outcome example (p = 0.5)
M = {M_L, M_H} with probabilities {0.5, 0.5}
⟨log(M)⟩ = 0.5 log(M_L) + 0.5 log(M_H)
⟨M⟩ = 0.5 M_L + 0.5 M_H
0
20
40
60
80
100
Money M
Why ⟨log(M)⟩ < log(⟨M⟩) ?
log(M) is concave (its slope decreases with M).
Gains when M is low increase utility a lot, while losses when M is high reduce utility only a little.
Averaging utility ⟨log(M)⟩ therefore gives a smaller value than utility at the average money (log(⟨M⟩)).
Economic Interpretation
People care about utility (happiness, satisfaction), not money itself.
With uncertainty in wealth, expected happiness is less than the happiness of expected wealth.
This is Jensen bias from the concavity of log(M).
Takeaway: For any concave function f, ⟨f(X)⟩ ≤ f(⟨X⟩), with equality only if X is certain (no uncertainty).
For U(M) = log(M), ⟨log(M)⟩ < log(⟨M⟩) whenever there is uncertainty in M.